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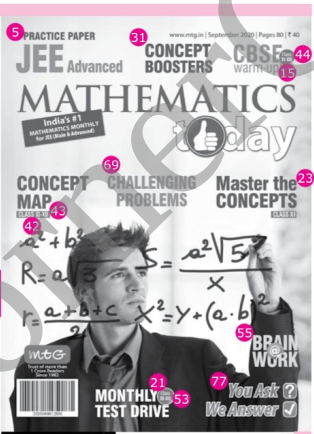

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Exam on 27th September

JEE 2020 PRACTICE PAPER ADVANCED

*ALOK KUMAR

SECTION-I

Single Correct Answer Type

1. Let $f(x) = \sin\theta(x^2 - 2)((\sin\theta)x + \cos\theta)$, $(\theta \neq m\pi, m \in I)$. Then $f(x)$ has
(a) local maxima at certain $x \in R^+$
(b) a local maxima at certain $x \in R^-$
(c) a local minima at certain $x = 0$
(d) a local minima at certain $x \in R^-$

2. If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\frac{19\pi}{8}\right)\right) = \frac{a\pi}{b}$
where a and b are in their lowest form, then $(a + b)$ equal to
(a) 17 (b) 20
(c) 23 (d) none of these

3. If a line with direction ratios 2 : 2 : 1 intersects the line $\frac{x-7}{3} = \frac{y-5}{2} = \frac{z-3}{1}$ and $\frac{x-1}{2} = \frac{y+1}{4} = \frac{z+1}{3}$ at A and B then $AB =$
(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) 3

4. Least value of modulus of slope of a line for which the line may touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{(a^3 + a^2 + a)^2} = 1$ is (a is non zero real)
(a) $\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

5. Length of latus rectum of ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$, if the normal at an end of latus rectum passes through one extremity of the minor axis, is
(a) $2 - \sqrt{3}$ (b) $6 - \sqrt{3}$
(c) $6 - 2\sqrt{3}$ (d) $6 - 2\sqrt{5}$

6. Radius of circle which is drawn on a normal chord of $y^2 = 4x$ as diameter and it passes through vertex is
(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $2\sqrt{3}$ (d) $3\sqrt{3}$

7. There is a point P on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ such that its distance to the right directrix is the average of its distance to the two foci. Let the x-coordinate of P be m/n with m and n being integers, $(n > 0)$ having no common factor except 1. Then $n - m$ equals
(a) 59 (b) 69 (c) –59 (d) –69

8. Suppose two complex numbers $z = a + ib$; $w = c + id$ satisfy the equation $\frac{z+w}{z} = \frac{w}{z+w}$ then
(a) both a and c are zeros
(b) both b and d are zeros
(c) both b and d must be non-zeros
(d) at least one of b and d is non-zero

9. The value of the expression $\sum_{0 \le i < j \le n} \sum (-1)^{i+j-1} {}^nC_i \cdot {}^nC_j$ is
(a) ${}^{2n-1}C_n$ (b) ${}^{2n}C_n$
(c) ${}^{2n+1}C_n$ (d) none of these

10. Let $P(x)$ be a polynomial with degree 2009 and leading co-efficient unity such that $P(0) = 2008$, $P(1) = 2007$, $P(2) = 2006, \ldots, P(2008) = 0$ then the value of $P(2009) + (n!) - a$, where n and a are natural numbers then value of $(n + a)$ is
(a) 2010 (b) 2009 (c) 2011 (d) 2008

11. Let $f(x) = x^2 + \lambda x + \mu\cos x$, λ being an integer and μ a real number. The number of ordered pairs (λ, μ) for which the equations $f(x) = 0$ and $f(f(x)) = 0$ have the same (non empty) set of real roots is

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(a) 4 (b) 6
(c) 8 (d) infinite

12. Mr. *A* makes a bet with Mr. *B* that in a single throw with two dice he will throw a total of seven before *B* throws four. Each of them has a pair of dice and they throw simultaneously until one of them wins, equal throws being disregarded. Probability that *B* wins, is
(a) 1/3 (b) 4/11 (c) 5/16 (d) 6/17

SECTION-II
Multiple Correct Answer Type

13. Consider the planes $P_1 : 2x + y + z + 4 = 0$
$P_2 : y - z + 4 = 0$ and $P_3 : 3x + 2y + z + 8 = 0$
Let L_1, L_2, L_3 be the lines of intersection of the planes P_2 and P_3, P_3 and P_1, and P_1 and P_2 respectively. Then,
(a) At least two of the lines L_1, L_2 and L_3 are non-parallel
(b) At least two of the lines L_1, L_2 and L_3 are parallel
(c) The three planes intersect in a line
(d) The three planes form a triangular prism

14. If two tangents can be drawn to the different branches of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (α, α^2), then
(a) $\alpha \in (-2, 0)$ (b) $\alpha \in (0, 2)$
(c) $\alpha \in (-\infty, -2)$ (d) $\alpha \in (2, \infty)$

15. The points on axis of parabola $3x^2 + 4x - 6y + 8 = 0$ from which three distinct normals can be drawn to it are
(a) $\left(\frac{-2}{3}, 2\right)$ (b) $\left(\frac{-2}{3}, 3\right)$
(c) $\left(\frac{-2}{3}, 4\right)$ (d) $\left(\frac{-2}{3}, 1\right)$

16. If three numbers are chosen randomly from the set $\{1, 3, 3^2, ..., 3^n\}$ without replacement, then the probability that they form an increasing geometric progression is
(a) $\frac{3}{2n}$ if n is odd (b) $\frac{3}{2n}$ if n is even
(c) $\frac{3n}{2(n^2-1)}$ if n is even (d) $\frac{3n}{2(n^2-1)}$ if n is odd

17. The number $\frac{5^k+3}{4}(k \in N)$, when divided by 10, may leave remainder
(a) 2 (b) 6 (c) 7 (d) 8

18. The values of a for which $\frac{x^3-6x^2+11x-6}{x^3+x^2-10x+8} + \frac{a}{30} = 0$ does not have a real solution is
(a) -10 (b) 12 (c) 5 (d) -30

19. If $A(z_1)$, $B(z_2)$, $C(z_3)$ are three points in argand plane where $|z_1 + z_2| = |z_1| - |z_2|$ and $|(1 - i)z_1 + iz_3| = |z_1| + |z_3 - z_1|$, then
(a) A, B and C lie on a fixed circle with centre $\left(\frac{z_2+z_3}{2}\right)$
(b) A, B, C form right angle triangle
(c) A, B, C form an equilateral triangle
(d) A, B, C form an obtuse angle triangle

20. If E and F are two independent events, such that $P(E \cap F) = \frac{1}{6}$, $P(E^C \cap F^C) = \frac{1}{3}$ and $(P(E) - P(F))(1 - P(F)) > 0$, then
(a) $P(E) = \frac{1}{2}$ (b) $P(E) = \frac{1}{4}$
(c) $P(F) = \frac{1}{3}$ (d) $P(F) = \frac{2}{3}$

21. The number of isosceles triangles with integer sides if no side exceeds 2008 is
(a) $(1004)^2$ if equal sides do not exceed 1004
(b) $2(1004)^2$ if equal sides exceed 1004
(c) $3(1004)^2$ if equal sides have any length ≤ 2008
(d) $(2008)^2$ if equal sides have any length ≤ 2008

SECTION-III
Comprehension Type

Paragraph for Question No. 22 to 23

Consider the planes $S_1 : 2x - y + z = 5$, $S_2 : x + 2y - z = 4$ having normals N_1 and N_2 respectively. $P(2, -1, 0)$ and $Q(1, 1, -1)$are points on S_1 and S_2 respectively.

22. A vector of magnitude $\sqrt{140}$ units and lies along the line of intersection of S_1 and S_2 is
(a) $2(5\hat{i} + 3\hat{j} - \hat{k})$ (b) $2(\hat{i} + 3\hat{j} + 5\hat{k})$
(c) $2\hat{i} - 6\hat{j} - 10\hat{k}$ (d) $2(3\hat{i} - \hat{j} + 5\hat{k})$

23. The distance of the origin from the plane passing through the point (1, 1, 1) and whose normal is perpendicular to N_1 and N_2 is
(a) $\frac{9}{\sqrt{61}}$ (b) $\frac{11}{\sqrt{35}}$ (c) $\frac{10}{\sqrt{61}}$ (d) $\frac{7}{\sqrt{35}}$

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Paragraph for Question No. 24 to 26

In a parallelogram $OABC$ with position vectors of A as $3\hat{i}+4\hat{j}$ and C as $4\hat{i}+3\hat{j}$ with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1. Also, the line segment AE intersects the line bisecting the $\angle AOC$ internally at P. CP when extended meets AB at point F.

24. The position vector of P is

(a) $\hat{i}+\hat{j}$ (b) $\frac{2}{3}(\hat{i}+\hat{j})$

(c) $\frac{13}{3}(\hat{i}+\hat{j})$ (d) $\frac{21}{5}(\hat{i}+\hat{j})$

25. The equation of line parallel to $\overline{CP}$ and passing through (2,3,4) is

(a) $\frac{x-2}{1}=\frac{y-3}{5}; z=4$ (b) $\frac{x-2}{1}=\frac{y-3}{6}; z=4$

(c) $\frac{x-2}{2}=\frac{y-3}{5}; z=3$ (d) $\frac{x-2}{3}=\frac{y-3}{5}; z=3$

26. The equation of plane containing line AC and at a maximum distance from B is

(a) $\vec{r}\cdot(\hat{i}+\hat{j})=7$ (b) $\vec{r}\cdot(\hat{i}-\hat{j})=7$

(c) $\vec{r}\cdot(2\hat{i}-\hat{j})=7$ (d) $\vec{r}\cdot(3\hat{i}+4\hat{j})=7$

Paragraph for Question No. 27 to 29

Consider the hyperbola $xy + x - y - 9 = 0$ and a line $x = 4$, which intersect the transverse axis at P. A parabola with vertex at P and axis parallel to y-axis passes through (1, 3), then

27. Equation of parabola is

(a) $y = x^2 - 2x + 4$ (b) $9y = x^2 - 8x + 20$

(c) $9y = x^2 - 8x + 34$ (d) $18y = x^2 - 8x + 20$

28. If tangents at P of parabola intersect hyperbola at P', then area of triangle PCP' (C is centre of hyperbola) is

(a) 1/2 sq. unit (b) 1 sq. unit

(c) 1/4 sq. unit (d) 2 sq. units

29. Equation of latus rectum of hyperbola which does not pass through IIIrd quadrant is

(a) $y + x - 2 = 0$ (b) $y + x + 1 = 0$

(c) $y + x + 4 = 0$ (d) $y + x - 8 = 0$

Paragraph for Question No. 30 to 32

Consider the parabolas $C_1 : y = x^2 + 1$ and $C_2 : x = y^2 + 1$. If PQ is the shortest distance and R, S are the points of contact of common tangent then

30. Distance of common tangent from origin is

(a) $\frac{3}{\sqrt{2}}$ (b) $\frac{3}{4}$ (c) $\frac{3}{4\sqrt{2}}$ (d) $\frac{3}{3\sqrt{3}}$

31. Length of PQ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{3}{2\sqrt{2}}$ (d) $\frac{1}{2}$

32. Area of region $PQRS$ is

(a) 1/4 sq. unit (b) 5/4 sq. units

(c) 7/4 sq. units (d) $5/\sqrt{2}$ sq. units

Paragraph for Question No. 33 to 35

$H : x^2 - y^2 = 9$, $P : y^2 = 4(x - 5)$, $L : x = 9$.

33. If L is a chord of contact of the hyperbola H, then the equation of the corresponding pair of tangents is

(a) $9x^2 - 8y^2 + 18x - 9 = 0$

(b) $9x^2 - 8y^2 - 18x + 9 = 0$

(c) $9x^2 - 8y^2 - 18x - 9 = 0$

(d) $9x^2 - 8y^2 + 18x + 9 = 0$

34. If R is the point of intersection of the tangents to H at the extremities of the chord L, then equation of the chord of contact of R with respect to the parabola P is

(a) $x = 7$ (b) $x = 9$

(c) $y = 7$ (d) $y = 9$

35. If the chord of contact of R (as in above question) with respect to the parabola P meets the parabola at T and T', S is the focus of the parabola, then area of the triangle STT' is equal to

(a) 8 sq. units (b) 9 sq. units

(c) 12 sq. units (d) 16 sq. units

Paragraph for Question No. 36 to 38

$P(a, 5a)$ and $Q(4a, a)$ are two points. Two circles are drawn through these points touching the axis of y.

36. Centre of these circles are at

(a) $(a, a), (2a, 3a)$ (b) $\left(\frac{205a}{18}, \frac{29a}{3}\right), \left(\frac{5a}{2}, 3a\right)$

(c) $\left(3a, \frac{29a}{3}\right), \left(\frac{205a}{9}, \frac{29a}{18}\right)$

(d) none of these

37. Angle of intersection of these circles is

(a) $\tan^{-1}(4/3)$ (b) $\tan^{-1}(40/9)$

(c) $\tan^{-1}(84/187)$ (d) $\pi/4$

38. If C_1, C_2 are the centres of these circles then area of ΔOC_1C_2, where O is the origin, is

(a) a^2 sq. units (b) $5a^2$ sq. units

(c) $10a^2$ sq. units (d) $20a^2$ sq. units

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SECTION-IV

Matrix Match Type

39. Match the following.

	Column I		Column II
(A)	The maximum value of $\sin(\cos x) + \cos(\sin x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is	(p)	$\cos(\cos 1)$
(B)	The minimum value of $\sin(\cos x) + \cos(\sin x)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is	(q)	$1 + \cos 1$
(C)	The maximum value of $\cos(\cos(\sin x))$ is	(r)	$\cos 1$
(D)	The minimum value of $\cos(\cos(\sin x))$ is	(s)	$1 + \sin 1$

40. In the column-I and column-II differential equations and their corresponding solutions are given. Match them.

	Column I		Column II
(A)	$(\cos x)\frac{dy}{dx} + y\sin x = 1$	(p)	$y = \frac{\sin x + c}{1 + x^2}$
(B)	$x\frac{dy}{dx} + y = x\sin x$	(q)	$y = \sin x + c\cos x$
(C)	$(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$	(r)	$xy = \sin x - x\cos x + c$
(D)	$\frac{dy}{dx} + (2\tan x)y = \sin x$	(s)	$y = \cos x + c\cos^2 x$

41. Match the following.

	Column I		Column II
(A)	The minimum value of ab if roots of the equation $x^3 - ax^2 + bx - 2 = 0$ are positive, is	(p)	24
(B)	The number of divisors of the form $12\lambda + 6(\lambda \in N)$ of the number 25200 are	(q)	3
(C)	The number of quadrilaterals formed in an octagon having two adjacent sides common with the polygon are	(r)	12
(D)	The number of solutions of the equation $[\cos x] = \tan x$, where $[\cdot]$ denotes greatest integer function $\forall\ x \in [0, 6\pi]$ are	(s)	18

42. Match the following.

	Column I		Column II
(A)	Number of ways to select n objects from $3n$ objects of which n are identical and rest are different is $k^{2k-1} + \frac{1}{k}\frac{(kn)!}{(n!)^2}$, k is	(p)	3
(B)	Number of interior points when diagonals of a convex polygon of n sides intersect if no three diagonal pass through the same interior point is ${}^{n}C_{\lambda}$, then λ is	(q)	2
(C)	Five digit numbers of different digit can be made in which digit are in descending order is ${}^{10}C_{\mu}$ then μ is	(r)	4
(D)	Number of terms in expansion of $(1 + 3^{1/3})^6$ which are free from radical sign is	(s)	5
		(t)	1

SECTION-V

Integer Answer Type

43. If length of the side BC of a ΔABC is 4 cm and $\angle BAC = 120°$, then the distance between incentre and excentre of the circle touching the side BC internally is

44. In parallelogram $ABCD$, P is any point inside parallelogram such that sum of area of ΔAPD and ΔBPC is 3 sq. units, then area of parallelogram $ABCD$ is

45. $A = (-3, 0)$ and $B = (3, 0)$ are the extremities of the base AB of triangle PAB. If the vertex P varies such that the internal bisector of angle APB of the triangle divides the opposite side AB into two segments AD and BD such that $AD : BD = 2 : 1$, then the maximum value of the length of the altitude of the triangle drawn through the vertex P is

46. Find the maximum value of $(\log_{3^{1/5}} a)\cdot(\log_{3^{1/2}} b)$. It is given that coefficient of 2nd, 3rd and 4th term in expansion of $(a + b)^n$ are in A.P. and the value of 3rd term is equal to 84 $(a, b > 1)$.

47. Let $S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \ldots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$, then find $|2000(S - 2000)|$.

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48. Consider two polynomials $f(x)$ and $g(x)$ as $g(x)=\sum_{r=0}^{200}\alpha_r x^r$ and $f(x)=\sum_{r=0}^{200}\beta_r x^r$. Given:

(i) $\beta_r = 1\ \forall\ r \geq 100$ (ii) $f(x+1)=g(x)$.

Let $A=\sum_{r=100}^{200}\alpha_r$. Find the remainder when A is divided by 15.

49. A sequence is obtained by deleting all perfect squares from set of natural numbers. The remainder when the 2003rd term of new sequence is divided by 2048, is

50. If $\alpha = e^{i2\pi/7}$ and $f(x)=A_0+\sum_{k=1}^{20}A_k x^k$ and the value of $f(x)+f(\alpha x)+f(\alpha^2 x)+\ldots+f(\alpha^6 x)$ is $k(A_0+A_7x^7+A_{14}x^{14})$, then find the value of k.

SOLUTIONS

1. **(b) :** $f(x)=(\sin^2\theta)x^3+\frac{1}{2}\sin 2\theta\cdot x^2-2\sin^2\theta\cdot x-\sin 2\theta$

$f'(x)=(3\sin^2\theta)x^2+x\sin 2\theta-2\sin^2\theta$

Then $D>0$ and product of roots <0

So $f(x)$ has local maxima at some $x\in R^-$

and local minima at some $x\in R^+$

2. **(b) :** On solving, we get $\frac{a\pi}{b}=\frac{13\pi}{7}\Rightarrow 13+7=20$.

3. **(d) :** $A(7+3\alpha, 5+2\alpha, 3+\alpha)$,

$B(1+2\beta, -1+4\beta, -1+3\beta)$

Dr's of AB are $2:2:1$

$$\frac{6+3\alpha-2\beta}{2}=\frac{3+\alpha-2\beta}{1}=\frac{4+\alpha-3\beta}{1}$$

$\alpha=-2,\ \beta=1$

$A(1, 1, 1),\quad B(3, 3, 2)$

$\therefore\ AB = 3$ units

4. **(d) :** Let slope of line is m

(asymptotes are given by $y=\pm\frac{b}{a}x$)

$\therefore\ a^2m^2-(a^3+a^2+a)^2\geq 0$

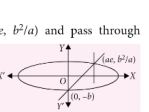
$\Rightarrow\ m^2\geq(a^2+a+1)^2$

$\Rightarrow\ m^2\geq\left(\frac{3}{4}\right)^2,\ |m|\geq\frac{3}{4}$

5. **(d) :** Let normal at $(ae,\ b^2/a)$ and pass through $(0, -b)$.

$\Rightarrow\ e^4+e^2-1=0$

$\Rightarrow\ e^2=\frac{\sqrt{5}-1}{2}$

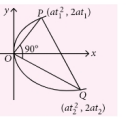


Also $b^2=a^2(1-e^2)$

$=4\left(1-\frac{\sqrt{5}-1}{2}\right)=2(3-\sqrt{5})$

Length of L.R. $=\frac{2b^2}{a}=\frac{4(3-\sqrt{5})}{2}=6-2\sqrt{5}$

6. **(d) :** Normal chord is PQ and normal is drawn at $P(at_1^2, 2at_1)$



$t_2=-t_1-\frac{2}{t_1}$.

Also $t_1t_2=-4$

$\Rightarrow\ -\frac{4}{t_1}=-t_1-\frac{2}{t_1}$

$\Rightarrow\ \frac{2}{t_1}=t_1\ \Rightarrow\ t_1=\sqrt{2}, t_2=-\sqrt{2}-\frac{2}{\sqrt{2}}$.

$P\equiv(2,\ 2\sqrt{2}),\quad Q\equiv(8,-4\sqrt{2})$

$PQ=\sqrt{36+72}=\sqrt{108},\ r=\frac{PQ}{2}=\sqrt{27}=3\sqrt{3}$

7. **(b) :** For the hyperbola, $e=\frac{5}{4}$

From the equation $2\left|x-\frac{a}{e}\right|=|ex+a|+|ex-a|$

It turns out that P has to be on the left branch. x-coordinate is found to be $-64/5$.

8. **(c) :** $(z+w)^2=zw\ \Rightarrow\ z^2+zw+w^2=0$

Let $\frac{z}{w}=t\ \Rightarrow\ \frac{z}{w}=\frac{-1\pm\sqrt{3}i}{2}$

$\arg z-\arg w=\frac{2\pi}{3}$ or $\arg z-\arg w=-\frac{2\pi}{3}$

9. **(a) :** Let the required value be S.

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(-1)^{i+j-1}\ {}^nC_i\,{}^nC_j=\sum_{i=0}^{n}(-1)^{2i-1}({}^nC_i)^2+2S$$

$$=-\sum_{i=0}^{n}({}^nC_i)^2+2S$$

$\Rightarrow\ S={}^{2n-1}C_{n-1}={}^{2n-1}C_n$

10. **(a) :** $P(x)-2008+x=x(x-1)(x-2)(x-3)$ ... $(x-2008)$

Put $x=2009$

$\therefore\ P(2009)+1=(2009)!$

11. **(a) :** Let α be a root of $f(x)$, so we have $f(\alpha)=0$ and thus $f(f(\alpha))=0$,

$\Rightarrow\ f(0)=0\ \Rightarrow\ \mu=0$

We then have $f(x)=x(x+\lambda)$ and thus $\alpha=0, -\lambda$.

$f(f(x))=x(x+\lambda)(x^2+\lambda x+\lambda)$

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We want λ such that $x^2 + \lambda x + \lambda$ has no real roots besides 0 and $-\lambda$. We can easily find that $0 \le \lambda < 4$.

12. (a) : We have $P(A) = P(T) = \frac{6}{36}$, $P(B) = P(4) = \frac{3}{36}$

Since equal throws are disregarded, hence in each throw A is twice as likely to win as B. Let $P(B) = p$, $P(A) = 2p$

Now, $3p = 1 \Rightarrow P(B) = 1/3$

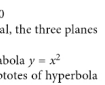
13. (a, c) : Observe that the lines L_1, L_2 and L_3 are parallel to the vector $(1, -1, -1)$.

Also $\Delta = 0 = \Delta_1$ and $b_1c_2 - b_2c_1 \neq 0$

$\because$ Coefficients are not proportional, the three planes intersect in a line

14. (c, d) : (α, α^2) lies on the parabola $y = x^2$

(α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ in 1st and 2nd quadrant



$\therefore$ Asymptotes are $y = \pm 2x$

$\therefore \; 2\alpha < \alpha^2 \Rightarrow \alpha < 0 \text{ or } \alpha > 2$

and $-2\alpha < \alpha^2$

$\alpha < -2 \text{ or } \alpha > 0$

$\therefore \; \alpha \in (-\infty, -2) \text{ or } \alpha \in (2, \infty)$

15. (a, b, c) : The parabola can be written as

$$\left(x + \frac{2}{3}\right)^2 = 2\left(y - \frac{10}{9}\right)$$

i.e. $X^2 = 2Y$ $\left(\text{where } X = x + \frac{2}{3}, Y = y - \frac{10}{9}\right)$.

A point on axis is $\left(\frac{-2}{3}, Y\right)$ from which three normals can be drawn if $Y > 0$.

16. (a, c) : Number of triplets $(3^r, 3^{r+1}, 3^{r+2})(0 \le r \le n-2)$ is $n-1$

Number of triplets $(3^r, 3^{r+2}, 3^{r+4})(0 \le r \le n-4)$ is $n-3$

Number of triplets $\left(3^r, 3^{r+\frac{n-1}{2}}, 3^{r+n-1}\right)(n \text{ odd})$ is 2

and number of triplets $\left(3^r, 3^{r+\frac{n}{2}}, 3^{r+n}\right)(n \text{ even})$ is 1

If n is odd, the number of favourable outcomes

$$= (n-1) + (n-3) + ... + 4 + 2 = \frac{n^2 - 1}{4}$$

and if n is even, the number of favourable outcomes

$$= (n-1) + (n-3) + ... + 3 + 1 = \frac{n^2}{4}$$

$\therefore$ Probability $= \frac{(n^2-1)/4}{{}^{(n+1)}C_3} = \frac{3}{2n}$ if n is odd

$= \frac{n^2/4}{{}^{(n+1)}C_3} = \frac{3n}{2(n^2-1)}$ if n is even

17. (a, c) : $\frac{5^k + 3}{4} = \frac{5^k - 5 + 8}{4} = \frac{5(5^{k-1} - 1)}{5 - 1} + 2$

$= 5 + 5^2 + 5^3 + ... (k-1) \text{ terms} + 2$

So, if $k - 1 =$ even, the last digit is 2

If $k - 1 =$ odd, the last digit is $5 + 2 = 7$.

18. (b, c, d) : $\frac{x^3 - 6x^2 + 11x - 6}{x^3 + x^2 - 10x + 8} = \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x+4)}$

$\therefore \quad x \neq 1, 2, -4$ then $f(x) = \frac{x-3}{x+4}$

Range of $f(x) = R - \left\{1, -\frac{2}{5}, -\frac{1}{6}\right\}$

So equation does not have a solution if $\frac{a}{30} = -1, \frac{2}{5}, \frac{1}{6}$

$\Rightarrow \; a = -30, 12, 5$

19. (a, b) : $\arg\frac{z_1}{z_2} = \pm\pi$ and

$|z_1 + i(z_3 - z_1)| = |z_1| + |z_3 - z_1|$ iff $\frac{z_1}{z_3 - z_1} = \frac{\pi}{2}$

Angle in a semicircle is 90^o

So centre of circle $= \left(\frac{z_2 + z_3}{2}\right)$ and ABC is right angle triangle.

20. (a, c) : $P(E \cap F) = P(E) \cdot P(F) = \frac{1}{6}$ ... (i)

$P(E^c \cap F^c) = (1 - P(E))(1 - P(F)) = \frac{1}{3}$

$\Rightarrow \quad P(E) + P(F) = \frac{5}{6}$ ... (ii)

$\Rightarrow \quad |P(E) - P(F)| = \frac{1}{6}$

As $(P(E) - P(F))(1 - P(F)) > 0$

$\Rightarrow \; P(E) > P(F) \Rightarrow P(E) - P(F) = \frac{1}{6}$ ... (iii)

Solving (ii) and (iii) $\Rightarrow \; P(E) = \frac{1}{2}, \; P(F) = \frac{1}{3}$

21. (a, b, c) : If the sides are a, a, b then the triangle is formed only when $2a > b$. So for any $a \in N$, b can change from 1 to $2a - 1$, where $a \le 1004$

$\Rightarrow$ no. of triangles $= 1 + 2 + 3 + ... + (2(1004) - 1)$

$= (1004)^2$

And if $1005 \le a \le 2008$, b can take any value from 1 to 2008

But a has 1004 possibilities hence

No. of triangles $= 1004 \times 2008 = 2(1004)^2$

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$\therefore$ Total no. of isosceles triangles = $3(1004)^2$

Remark : The above count of isosceles triangle also includes equilateral triangle.

(22-23):

Unit vector along line of intersection of S_1 and S_2

$$= \pm \frac{(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}-\hat{k})}{\left|(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+2\hat{j}-\hat{k})\right|} = \pm\frac{(-\hat{i}+3\hat{j}+5\hat{k})}{\sqrt{35}}$$

22. (c) : $\pm 2\sqrt{35} \times \dfrac{(-\hat{i}+3\hat{j}+5\hat{k})}{\sqrt{35}} = \pm 2(-\hat{i}+3\hat{j}+5\hat{k})$

23. (d) : Equation of plane is

$-1(x-1)+3(y-1)+5(z-1)=0$

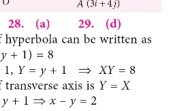
$$\text{Distance} = S = \left|\frac{1-3-5}{\sqrt{35}}\right| = \frac{7}{\sqrt{35}}$$

24. (d) : $\overrightarrow{OB} = 7\hat{i}+7\hat{j},\ \overrightarrow{OE} = 5\hat{i}+\frac{13}{3}\hat{j},\ \overrightarrow{OP} = \frac{21}{5}(\hat{i}+\hat{j})$

25. (b) : Direction ratios of CP are proportional to (1, 6, 0) then equation of line passing through (2, 3, 4) and parallel to CP is $\dfrac{x-2}{1} = \dfrac{y-3}{6} = \dfrac{z-4}{0}$

26. (a) : The plane containing line AC and at a maximum distance from B must be perpendicular to the plane $OABC$. Since $OABC$ is rhombus, so $\overrightarrow{OB}$ must normal to the plane. So equation of required plane is

$$[\vec{r}-4\hat{i}-3\hat{j}]\cdot(\hat{i}+\hat{j})=0 \Rightarrow \vec{r}\cdot(\hat{i}+\hat{j})=7$$



27. (c) **28. (a)** **29. (d)**

Equation of hyperbola can be written as

$(x-1)(y+1)=8$

Let $X = x-1,\ Y = y+1 \Rightarrow XY = 8$

Equation of transverse axis is $Y = X$

$\Rightarrow x-1 = y+1 \Rightarrow x-y=2$

$\therefore\ P \equiv (4, 2)$

Let equation of parabola be $y = ax^2 + bx + c$

$2 = 16a + 4b + c$ ... (1)

Also $\left.\dfrac{dy}{dx}\right|_{(4,2)} = 0 \Rightarrow 8a + b = 0$ ... (2)

Parabola passes through (1, 3)

$\Rightarrow 3 = a + b + c$ ... (3)

Solving (1), (2), (3) we get $a = \dfrac{1}{9}, b = -\dfrac{8}{9}, c = \dfrac{34}{9}$

So parabola is $9y = x^2 - 8x + 34$

Now tangent at P is $y = 2$

P' is $\left(\frac{11}{3}, 2\right)$, C is $(1, -1)$

$$A = \frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 11/3 & 2 & 1 \end{vmatrix} = \frac{1}{2}\left[1(2-2)+1\left(4-\frac{11}{3}\right)+\left(8-\frac{22}{3}\right)\right]$$

$= 1/2$ sq. unit.

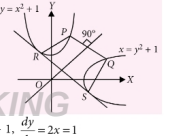
Let latus rectum is $y + x + \lambda = 0$

Now length of $\perp$ from centre $= 4\sqrt{2}$

So, $\dfrac{|\lambda|}{\sqrt{2}} = 4\sqrt{2} \Rightarrow \lambda = \pm 8$

So equation of latus rectum is $y + x - 8 = 0$.

30. (c) : PQ is common normal



$y = x^2 + 1,\ \dfrac{dy}{dx} = 2x = 1$

$\therefore\ x = \dfrac{1}{2},\ y = \dfrac{5}{4}$

As PQ is the shortest distance occur along common normal.

So, we can calculate P and Q

$$P \equiv \left(\frac{1}{2}, \frac{5}{4}\right),\ Q \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$$

Slope of common tangent is -1

Let equation of C.T is $y = -x + c$

Also $x^2 + x + 1 - c = 0$

$D = 0 \Rightarrow 1 - 4(1 - c) = 0,\ c = 3/4$

$\therefore$ Common tangent is $y = -x + \dfrac{3}{4} \Rightarrow 4x + 4y = 3$

Distance of common tangent from origin

$$p = \frac{|-3|}{\sqrt{16+16}} = \frac{3}{4\sqrt{2}}$$

31. (c) : $PQ = \sqrt{\frac{9}{16} + \frac{9}{16}} = \frac{3}{2\sqrt{2}}$

32. (b) : Also $R \equiv \left(-\frac{1}{2}, \frac{5}{4}\right),\ S \equiv \left(\frac{5}{4}, -\frac{1}{2}\right)$

$RS = \dfrac{7}{2\sqrt{2}}$, Distance between PQ and RS, $(d) = \dfrac{1}{\sqrt{2}}$.

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Area of $PQRS = \frac{1}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \times \frac{1}{2} = \frac{5}{4}$ sq. units

33. (b) : Let $R(h, k)$ be the point of intersection of the tangents to H at the extremities of the chord $L : x = 9$ then equation of L is $hx - ky = 9$

$\Rightarrow \quad h = 1, \; k = 0.$

$\therefore$ Coordinates of R are $(1, 0)$.

Equation of the pair of tangents from R to H is

$(x^2 - y^2 - 9)(1 - 9) = (x - 9)^2 \qquad (SS_1 = T^2)$

$\Rightarrow \; 9x^2 - 8y^2 - 18x + 9 = 0.$

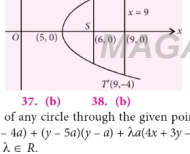
34. (b) : Now equation of the chord of contact of $R(1, 0)$ with respect to the parabola P: $y^2 = 4(x - 5)$ is

$y \times 0 = 2(x + 1) - 20 \;\Rightarrow\; x = 9$

35. (c) : Coordinates of T and T' are $(9, 4), (9, -4)$.

Coordinates of focus S of P are $(6, 0)$.

Area of $\Delta STT' = 4 \times 3 = 12$ sq. units.



36. (b) **37. (b)** **38. (b)**

Equation of any circle through the given points is

$(x - a)(x - 4a) + (y - 5a)(y - a) + \lambda a(4x + 3y - 19a) = 0$

for some $\lambda \in R$.

As it touches the y-axis, $\left(-3a + \frac{3a\lambda}{2}\right)^2 = 9a^2 - 19\lambda a^2$

Solving $\lambda = 0, \; \frac{-40}{9}$

The required circles are $x^2 + y^2 - 5ax - 6ay + 9a^2 = 0$,

$x^2 + y^2 - 5ax - 6ay + 9a^2 - \frac{40a}{9}(4x + 3y - 19a) = 0$

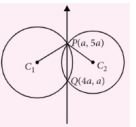
Hence centres are $\left(\frac{5a}{2}, 3a\right)$ and $\left(\frac{205a}{18}, \frac{29a}{3}\right)$

The centres of the given circles are

$C_1\left(\frac{205a}{18}, \frac{29a}{3}\right)$

and $C_2\left(\frac{5a}{2}, 3a\right)$

Now the angle of intersection θ of these two circles is the angle between the radius vectors at the common point P to the two circles *i.e.*, $\angle C_1PC_2 = \theta$



Slope of $C_1P = \frac{\frac{29}{3}a - 5a}{\frac{205}{18}a - a} = \frac{84}{187}$

and slope of $C_2P = \frac{5a - 3a}{a - \frac{5a}{2}} = -\frac{4}{3}$

Now, $\tan\theta = \left|\frac{\frac{84}{187} + \frac{4}{3}}{1 - \frac{84}{187} \times \frac{4}{3}}\right| = \frac{252 + 748}{561 - 336} = \frac{40}{9}$

$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{40}{9}\right)$

Area of $\Delta OC_1C_2 = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{205}{18} & \frac{29}{3} & 1 \\ \frac{5}{2} & 3 & 1 \end{vmatrix} a^2 = 5a^2$ sq. units

39. (A) → (s); (B) → (r); (C) → (p); (D) → (r)

Let $f(x) = \sin(\cos x) + \cos(\sin x)$

f is an even function. We can take $x \in \left[0, \frac{\pi}{2}\right]$.

In $\left[0, \frac{\pi}{2}\right]$, $\sin x$ is increasing and $\cos x$ is decreasing.

Hence f is a decreasing function. Therefore, maximum value of f is $f(0) = \sin 1 + 1$ and minimum value is $f(\pi/2) = 0 + \cos 1$.

Let $g(x) = \cos(\cos(\sin x))$. Obviously g is an even periodic function of period π. Hence g takes all of its values for $x \in \left[0, \frac{\pi}{2}\right]$.

It can be seen that g is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

So maximum value of g is $g(\pi/2) = \cos(\cos 1)$, and minimum value of g is $g(0) = \cos 1$.

40. (A) → (q); (B) → (r); (C) → (p); (D) → (s)

(A) $\frac{dy}{dx} + (\tan x)y = \sec x$

$\therefore \quad \text{I.F.} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

$y \sec x = \int \sec^2 x dx = \tan x + c$

$\Rightarrow \quad y = \sin x + c \cos x$

(B) $\frac{dy}{dx} + \frac{y}{x} = \sin x$, I.F. $= x$

$yx = \int x \sin x dx + c = \sin x - x \cos x + c$

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(C) $\dfrac{dy}{dx}+\left(\dfrac{2x}{1+x^2}\right)y=\dfrac{\cos x}{1+x^2}$

$\therefore$ I.F. $= 1+x^2 \Rightarrow y(1+x^2)=\int \cos x\ dx=\sin x+c$

(D) $\dfrac{dy}{dx}+(2\tan x)y=\sin x$

$\therefore$ I.F. $=\sec^2 x$

$y\sec^2 x=\int \sin x\sec^2 x\,dx=\sec x+c$

$\Rightarrow\ y=\cos x+c\cos^2 x$

41. (A) → (s); (B) → (r); (C) → (p); (D) → (q)

(A) $x^3-ax^2+bx-2=0$

Let x_1, x_2, x_3 be roots

A.M. $\geq$ G.M.

$\Rightarrow \left(\dfrac{x_1+x_2+x_3}{3}\right)\left(\dfrac{\Sigma x_1x_2}{3}\right)\geq (x_1x_2x_3)^{1/3}(x_1x_2x_3)^{2/3}$

$\Rightarrow \dfrac{ab}{9}\geq 2\ \Rightarrow (ab)_{\min}=18$

(B) $25200=2^4 3^2 5^2 7^1$

Now, if divisor is of the form $12\lambda+6$ *i.e.* $6(2\lambda+1)$ then there must be exactly one 2 and at least one 3.

$\therefore$ Number of divisors are $1\cdot2\cdot(2+1)(1+1)=12$

(C) Number of quadrilaterals are $n(n-5)=8\times3=24$

(D) $|\cos x|=\tan x$

$\tan x=-1$ and $\cos x<0$

$\therefore$ Three solutions.

42. (A) → (q); (B) → (r); (C) → (s); (D) → (p)

(A) Required number of selection

$= {}^{2n}C_0+{}^{2n}C_1+...+{}^{2n}C_n=2^{2n-1}+\dfrac{1}{2}\dfrac{(2n)!}{(n!)^2}$

(B) ${}^nC_4$ (each quadrilateral gives one point of intersection)

(C) $x_4>x_3>x_2>x_1>x_0$

${}^{10}C_5$ (5 distinct digits selection)

(D) Terms involving 3^0, 3^1, $3^2 \rightarrow 3$

43. (8) : I_1 and I are excentre and incentre of $\Delta\Delta BC$

$II_1=AI_1-AI$

$=(r_1-r)\operatorname{cosec}\dfrac{A}{2}=a\tan\dfrac{A}{2}\operatorname{cosec}\dfrac{A}{2}=\dfrac{a}{\cos(A/2)}=8$

44. (6) : Sum of area of ΔAPD and ΔBPC is equal to 1/2 of area of parallelogram $ABCD$.

45. (4) : The point E dividing $\overline{AB}$ externally in the ratio 2 : 1 is (9 , 0). Since P lies on the circle described on $\overline{DE}$ as diameter, coordinates of P are of the form $(5+4\cos\theta, 4\sin\theta)$

$\therefore$ maximum length of the altitude drawn from P to the base $AB=|4\sin\theta|_{\max}=4$.

46. (1) : In expansion of $(a+b)^n$ the coefficient of 2nd, 3rd and 4th term are in A.P. which gives $n=7$

Also ${}^7C_2\ a^5b^2=84 \Rightarrow a^5b^2=4$

Now $\dfrac{\log_2 a^5+\log_2 b^2}{2}\geq(\log_2 a^5\cdot\log_2 b^2)^{1/2}$

$\Rightarrow\ k\leq\left(\dfrac{\log_2 a^5b^2}{2}\right)^2$, where $k=\log_2 a^5\cdot\log_2 b^2$

$\Rightarrow\ k\leq 1\ \Rightarrow$ maximum value of k is 1.

47. (1) : $t_r=\sqrt{1+\dfrac{1}{r^2}+\dfrac{1}{(r+1)^2}}=\sqrt{\dfrac{r^2+(r+1)^2+r^2(r+1)^2}{r^2(r+1)^2}}$

$=\sqrt{\dfrac{2r^2+2r+1+r^2(r^2+2r+1)}{r^2(r+1)^2}}$

$=\sqrt{\dfrac{r^4+2r^3+3r^2+2r+1}{r^2(r+1)^2}}=\dfrac{r^2+r+1}{r(r+1)}=\dfrac{1}{r(r+1)}+1$

$=1+\dfrac{1}{r}-\dfrac{1}{r+1}$, $S=2000-\dfrac{1}{2000}$, $\lfloor 2000(S-2000)\rfloor=1$

48. (1) : $\sum_{r=0}^{200}\alpha_r x^r=\sum_{r=0}^{200}\beta_r(1+x)^r$

$\alpha_0+\alpha_1x+\alpha_2x^2+....+\alpha_{200}x^{200}$

$=\beta_0+\beta_1(1+x)+....+\beta_{200}(1+x)^{200}$

Equating coefficient of x^{100}, we get

$\alpha_{100}={}^{100}C_{100}+{}^{101}C_{100}+.....{}^{200}C_{100}={}^{201}C_{101}$

Similarly we can find α_{100}, ..., α_{200}

$\sum_{r=100}^{200}\alpha_r={}^{201}C_{101}+{}^{201}C_{102}+.....{}^{201}C_{201}$

$A=2^{100}(2^{101}-1)$

When A is divided by 15 remainder is 1.

49. (0) : Since $\left[\sqrt{2046}\right]=\left[\sqrt{2047}\right]=\left[\sqrt{2048}\right]$

$=\left[\sqrt{2049}\right]=45$

$\therefore$ 2003^{rd} term is $2003+45=2048$

Hence remainder is 0

Remark : There is no perfect square between the numbers m^2 and $(m+1)^2$.

50. (7) : $f(x)+f(\alpha x)+f(\alpha^2x)+.....+f(\alpha^6x)$

$=7A_0+\sum_{k=1}^{20}A_kx^k(1+\alpha^k+.....+\alpha^{6k})$

but when $k\neq7$ and $k\neq14$, then

$1+\alpha^k+\alpha^{2k}+....+\alpha^{6k}=0$

Hence $f(x)+f(\alpha x)+...+f(\alpha^6x)$

$=7A_0+7A_7x^7+7A_{14}x^{14}=7(A_0+A_7x^7+A_{14}x^{14})$

$\Rightarrow\ k=7$

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CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 4

Permutations and Combinations

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long answer-**II** type questions carrying **6** marks each.

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SECTION - A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Find the value of n such that $\frac{{}^{n}P_4}{{}^{n-1}P_4} = \frac{5}{3}$, $n > 4$.
(a) 11 (b) 10 (c) 13 (d) 12

2. If eleven members of a committee sit at a round table so that the President and Secretary always sit together, then the number of arrangements is
(a) $10! \times 2$ (b) 10! (c) $9! \times 2$ (d) $11! \times 2!$

3. Find the number of permutations of the letters of the word ALLAHABAD.
(a) 7530 (b) 7540 (c) 7560 (d) 7500

4. The number of ways you can find to pack 9 different books into five parcels if four of the parcels must contain two books each, is
(a) 945
(b) ${}^{9}C_2 \times {}^{7}C_2 \times {}^{5}C_2 \times {}^{3}C_2$
(c) ${}^{9}C_2 \times {}^{7}C_2 \times {}^{5}C_2 \times {}^{3}C_2 \times 5!$
(d) None of these

5. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$, then
(a) $n > 5$ (b) $n > 6$
(c) $n > 7$ (d) None of these

6. In how many ways a committee consisting of 3 men and 2 women, can be chosen from 7 men and 5 women?
(a) 45 (b) 350 (c) 4200 (d) 230

7. The triplet (x, y, z) is chosen from the set $\{1, 2, 3, \ldots n\}$, such that $x \le y < z$. The number of such triplets is
(a) n^3 (b) ${}^{n}C_3$
(c) ${}^{n}C_2$ (d) ${}^{n}C_2 + {}^{n}C_3$

8. ABC is a triangle. 4, 5, 6 points are marked on the sides AB, BC, CA respectively, the number of triangles having vertices on different sides is
(a) $(4 + 5 + 6)!$ (b) $(4 - 1)(5 - 1)(6 - 1)$
(c) $5!\ 4!\ 6!$ (d) $4 \times 5 \times 6$

9. There are four balls of different colours and four boxes of colours same as those of the balls. The

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number of ways in which the balls, one in each box, could be placed such that a ball does not go to box of its own colour, is
(a) 8 (b) 7 (c) 9 (d) 10

10. Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.
(a) 22 (b) 24 (c) 26 (d) 28

(Q. 11-Q.15) Fill in the blanks.

11. If $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!}$, then the value of x is _____ .

12. Number of ways in which different signals that can be made with 5 flags from 8 flags of different colours is _____ .

OR

If ${}^{n}C_{5}={}^{n}C_{7}$, then the value of n is _____ .

13. If ${}^{10}C_{x}={}^{10}C_{x+4}$, then value of x is _____ .

14. A crocodile is known to have not more than 68 teeth. The total number of crocodiles with different set of teeth, are _____ .

OR

The number of six digit numbers, whose all digits are odd (*i.e.*, 1, 3, 5, 7, 9), is _____ .

15. There were two women participants in a chess tournament. The number of games the men played between themselves exceeded by 52 the number of games they played with women. If each player played one game with each other, the number of men in the tournament, was _____ .

(Q. 16-Q.20) Answer the following questions.

16. All the letters of the word 'EAMCOT' are arranged in different possible ways. Find the number of ways in which no two vowels are adjacent to each other.

17. Evaluate :
(i) 7! (ii) 4! – 3!

18. Determine n if $C(2n, 3) : C(n, 3) = 11 : 1$.

OR

How many words, with or without meaning, can be formed using all the letters of the word 'EQUATION', using each letter exactly once?

19. Show that : $\frac{12!}{6!}=1\cdot3\cdot5\cdot7\cdot9\cdot11\cdot2^{6}$.

20. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated?

SECTION - B

21. Find all pairs of consecutive even positive integers, both of which are larger than 5 such that their sum is less than 23.

22. How many words (with or without meaning) of three distinct English alphabets are there?

OR

How many words, each of 3 vowels and 2 consonants, can be formed from the letters of the word 'INVOLUTE'?

23. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of one man and two women?

24. In how many ways we can select a cricket team of eleven from 17 players in which only 5 players can bowl if each cricket team of 11 must include exactly 4 bowlers?

OR

If $\frac{n!}{5!(n-1)!}$ and $\frac{n!}{7!(n-3)!}$ are in the ratio 21 : 1, find the value of n.

25. A convex polygon has 65 diagonals. Find the number of sides of the polygon.

26. Determine the number of ways of choosing 5 cards out of a deck of 52 cards which include exactly one ace.

SECTION - C

27. How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 assuming that :
(i) Repetition of the digits is allowed?
(ii) Repetition of the digits is not allowed?

28. If all the letters of the word 'MOTHER' are written in all possible orders and the words so formed are arranged in a dictionary order, then find the rank of the word 'MOTHER'.

OR

Eighteen guests are to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on other side of the table. Find the number of possible seating arrangements.

29. From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them join or none of them

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will join. In how many ways can the excursion party be chosen?

30. Find the number of words with or without meaning, which can be made using all the letters of the word 'AGAIN'. If these words are written as in a dictionary, what will be the 50^{th} word?

31. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when committee consist of

(i) atleast 3 girls (ii) at most 3 girls?

OR

How many numbers greater than 1000000 can be formed by using digits 1, 2, 0, 2, 4, 2, 4?

32. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, find the values of n and r.

SECTION - D

33. (i) How many different words can be formed with the letters of the word HARYANA?

(ii) How many of these begin with H and end with N?

(iii) In how many of these H and N are together?

34. In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats?

OR

Prove that : $^{2n}C_n = \dfrac{2^n[1\cdot 3\cdot 5 \ldots (2n-1)]}{n!}$

35. A question paper contains 12 questions, divided into three parts. Part *A* contains 6 questions while *B* and *C* contains 3 questions each. A candidate is required to attempt 6 questions selecting at least two from part *A* and at least one from each of part *B* and part *C*. In how many ways can the candidate select 6 questions?

OR

How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?

36. A mathematics paper consists of 10 questions divided into two parts I and II, each part containing 5 questions. A student is required to attempt 6 questions in all, taking atleast 2 question from each part. In how many ways the student select the questions?

SOLUTIONS

1. **(b)** : Given that $\dfrac{{}^nP_4}{{}^{n-1}P_4} = \dfrac{5}{3}$

$\Rightarrow \quad 3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$

$\Rightarrow \quad 3n = 5(n-4)$ [as $(n-1)(n-2)(n-3) \neq 0, n > 4$]

$\Rightarrow \quad n = 10$

2. **(c)** : Since, out of eleven members, two members sit together, therefore the number of arrangements $= 9! \times 2$ ($\because$ two members can sit in two ways.)

3. **(c)** : There are 9 letters in which there are 4A's, 2L's and rest all are different.

Therefore, the required number of arrangements

$= \dfrac{9!}{4!\ 2!} = \dfrac{5\times 6\times 7\times 8\times 9}{2} = 7560$

4. **(a)** : Here 9 things are to be divided into 5 groups out of which four groups have equal number of elements, *viz*. 2 and one group has only one element.

$\therefore$ Required number of ways $= \dfrac{9!}{2!2!2!2!4!} = 945.$

5. **(c)** : We have, ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$

$\Rightarrow \quad {}^nC_4 > {}^nC_3 \Rightarrow \dfrac{n!}{(n-4)!4!} > \dfrac{n!}{(n-3)!3!}$

$\Rightarrow \quad \dfrac{1}{4} > \dfrac{1}{n-3} \Rightarrow n-3 > 4 \Rightarrow n > 7.$

6. **(b)** : Out of 7 men, 3 men can be chosen in ${}^7C_3$ ways and out of 5 women, 2 women can be chosen in ${}^5C_2$ ways. Hence, the committee can be chosen in ${}^7C_3 \times {}^5C_2 = 350$ ways.

7. **(d)** : Number of selections when $x < y < z$ is ${}^nC_3$. Number of selections when $x = y < z$ is ${}^nC_2$.

$\therefore$ Required number $= {}^nC_3 + {}^nC_2$

8. **(d)** : Required number of such triangles

$= {}^4C_1 \times {}^5C_1 \times {}^6C_1 = 4 \times 5 \times 6$

9. **(c)** : The number of ways in which four different balls can be placed in four different boxes

$= {}^4C_1 + {}^3C_1 + {}^2C_1 + {}^1C_1 = 4 + 3 + 2 + 1 = 10$

$\therefore$ Required number of ways $= 10 - 1 = 9$

(since, only one way in which the same ball have a same box).

Monthly Test Drive-4 CLASS XI — ANSWER KEY

1.	(a)	**2.**	(b)	**3.**	(d)	**4.**	(b)	**5.**	(d)
6.	(a)	**7.**	(a, b, c)	**8.**	(a, b, c)	**9.**	(a, c)	**10.**	(a)
11.	(a)	**12.**	(a, b, c, d)	**13.**	(a, c)	**14.**	(c)	**15.**	(d)
16.	(d)	**17.**	(7)	**18.**	(9)	**19.**	(2)	**20.**	(7)

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10. (b) : There will be as many words as there are ways of filling 4 vacant places □ □ □ □ in succession by the 4 letters, keeping in mind that the repetition is not allowed. Thus, the number of ways in which the 4 places can be filled, by the multiplication principle, is $4 \times 3 \times 2 \times 1 = 24$. Hence, the required number of words is 24.

11. Given, $\frac{1}{6!}+\frac{1}{7!}=\frac{x}{8!} \Rightarrow \frac{7!+6!}{6!7!}=\frac{x}{8!}$

$\Rightarrow \quad \frac{6!(7+1)}{6!}=\frac{x}{8} \Rightarrow 8=\frac{x}{8} \Rightarrow x=64$

12. The number of ways taking 5 flags out of 8 flags

$= {}^8P_5 = \frac{8!}{(8-5)!} = \frac{8!}{3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3!}{3!} = 6720.$

OR

Given, ${}^nC_5 = {}^nC_7 \Rightarrow n = 5+7 = 12$

13. Given, ${}^{10}C_x = {}^{10}C_{x+4}$

$\Rightarrow \quad 10 = x + x + 4 \Rightarrow 6 = 2x \Rightarrow x = 3.$

14. The required number is equal to the number of all the subsets of a 68 elements set *i.e.* ${}^{68}C_0 + {}^{68}C_1 + {}^{68}C_2 + \ldots + {}^{68}C_{68} = 2^{68} = 16^{17}$.

MAGAZINE KING

OR

Since six digit numbers, whose all digits are odd, are to be formed suggests that repetition of digits is must.

$\because$ Available digits are 1, 3, 5, 7, 9

Hence required numbers is 5^6.

15. Let there were n men playing in the tournament. Then, according to the given condition, we have

${}^nC_2 - {}^nC_1 \times {}^2C_1 = 52 \Rightarrow \frac{n(n-1)}{2} - 2n = 52$

$\Rightarrow \quad n^2 - n - 4n = 104 \Rightarrow n^2 - 5n - 104 = 0 \Rightarrow n = 13.$

16. The total number of words formed by using all the 6 letters of the word 'EAMCOT' in ${}^6P_6 = 6! = 720$

So, the total number of words in which no two vowels are adjacent to each other = Total number of words – number of words in which vowels are always together

$= 720 - 4!\,3! = 720 - 144 = 576$

17. (i) $7! = 7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 5040$

(ii) $4! - 3! = 3!\,(4-1) = 3 \times 3 \times 2 = 18$

18. Here, $C(2n, 3) : C(n, 3) = 11 : 1$

$\Rightarrow \quad {}^{2n}C_3 : {}^nC_3 = 11 : 1$

$\Rightarrow \quad \frac{2n!}{3!(2n-3)!} \times \frac{3!(n-3)!}{n!} = \frac{11}{1}$

$\Rightarrow \quad \frac{(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = \frac{11}{1}$

$\Rightarrow \quad \frac{(2n)(2n-1)(2)(n-1)}{n(n-1)(n-2)} = \frac{11}{1}$

$\Rightarrow \quad 4\,(2n-1) = 11\,(n-2)$

$\Rightarrow \quad 8n - 4 = 11n - 22 \Rightarrow 3n = 18 \Rightarrow n = 6.$

OR

No. of different letters in EQUATION = 8

Number of letters to be taken at a time = 8

If P is the number of words thus formed, then

$P = P(8, 8) = \frac{8!}{(8-8)!} = 8! = 40320$

19. $\frac{12!}{6!} = \frac{12\times11\times10\times9\times8\times7\times6!}{6!}$

$= 12 \times 11 \times 10 \times 9 \times 8 \times 7 = 1 \times 3 \times 5 \times 7 \times 9 \times 11 \times 2^6$

20. 3-digit numbers are to be formed using digits 1 to 9 with no digit repeated.

This can be done in $9 \times 8 \times 7 = 504$ ways.

21. Let the consecutive even positive integers be $2n$, $2n+2$

According to question, $2n > 5 \Rightarrow n > \frac{5}{2}$

and $2n + 2n + 2 < 23 \Rightarrow 4n < 21 \Rightarrow n < \frac{21}{4}$

$\therefore$ Possible values of n are 3, 4, 5.

Hence, possible pairs of consecutive even positive integers (6, 8), (8, 10), (10, 12).

22. There are 26 distinct English alphabets.

First alphabet can be chosen in 26 ways.

Second alphabet can be chosen in 25 ways.

Third alphabet can be chosen in 24 ways.

$\therefore$ Total number of three letter words

$= 26 \times 25 \times 24 = 15600$

OR

Number of letters in the word = 8

Number of vowels in the word = 4 (I, O, U, E)

Number of consonants in the word = 4 (N, V, L, T)

Out of 4 vowels, we have to select 3.

Out of 4 consonants, we have to select 2.

Also, we have to arrange 3 vowels and 2 consonants.

$\therefore$ Required no. of words = $C(4, 3) \cdot C(4, 2) \cdot 5!$

$= 4 \times \frac{4\times3}{2} \times 120 = 2880$

23. There are 2 men and 3 women in the group.

A committee of 3 persons can be selected in

${}^5C_3 = 10$ ways

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A committee consisting of 1 man and 2 women
$= {}^2C_1 \times {}^3C_2 = 2 \times 3 = 6$ ways.

24. Total players = 17, Bowlers = 5,
Non-bowlers = 12
Team of 11 players is to be selected having exactly 4 bowlers
$\therefore$ No. of ways $= {}^5C_4 \times {}^{12}C_7 = 5 \times 792 = 3960$

OR

Here, $\frac{n!}{5!(n-1)!} : \frac{n!}{7!(n-3)!} = 21:1$

$\Rightarrow \frac{n!}{5!(n-1)!} \times \frac{7!(n-3)!}{n!} = \frac{21}{1}$

$\Rightarrow \frac{7!(n-3)!}{5!(n-1)!} = \frac{21}{1} \Rightarrow \frac{7\times6\times5!(n-3)!}{5!(n-1)(n-2)(n-3)!} = \frac{21}{1}$

$\Rightarrow \frac{42}{n^2-3n+2} = \frac{21}{1}$

$\Rightarrow n(n-3) = 0 \Rightarrow n = 0$ or $n = 3$
But when $n = 0$ then $(n-1)!$ and $(n-3)!$ are not defined.
Thus, $n = 3$

25. Let number of sides or vertices be n
$\therefore$ Number of diagonals $= {}^nC_2 - n$

$\Rightarrow 65 = {}^nC_2 - n \Rightarrow 65 = \frac{n(n-1)}{2} - n$

$\Rightarrow n^2 - 3n = 130 \Rightarrow (n-13)(n+10) = 0$
$\Rightarrow n = 13$ or $n = -10$ (Rejected)

26. Here, total number of cards = 52
We have to choose 5 cards, which include 1 ace and other 4 cards. *i.e.*, we have to select 1 ace out of 4 aces and 4 others out of 48 others cards.
$\therefore$ Total number of ways of choosing the cards
$= {}^4C_1 \times {}^{48}C_4$

$= \frac{4!}{3!1!} \times \frac{48!}{4!44!} = \frac{4\times3!}{3!} \times \frac{48\times47\times46\times45\times44!}{4!44!}$

$= 4 \times \frac{48\times47\times46\times45}{24} = 778320$

27. (i) There are five digits, 1, 2, 3, 4 and 5. Each digit can be selected any number of times. Hence, we can select first digit in 5 ways. The second digit and third digit can also be selected in 5 ways.
$\therefore$ The number of ways in which the selection of three digits can be made $= 5 \times 5 \times 5 = 125$.
(ii) Under the restriction, first digit can be selected in 5 ways. After the selection of first digit four digits are left. Second digit can be selected in 4 ways and third digit can be selected in 3 ways.
$\therefore$ Required total no. of ways $= 5 \times 4 \times 3 = 60$

28. There are 6 letters in the word 'MOTHER'
Hence, total no. of words starting with E = 5!
Total no. of words starting with H = 5!
Total no. of words starting with ME = 4!
Total no. of words starting with MH = 4!
Total no. of words starting with MOE = 3!
Total no. of words starting with MOH = 3!
Total no. of words starting with MOR = 3!
Total no. of words starting with MOTE = 2!
Thus, next word formed is MOTHER
Total no. of words before MOTHER
$= 2 \times 5! + 2 \times 4! + 3 \times 3! + 2! = 308$
Rank of the word MOTHER $= 308 + 1 = 309^{th}$

OR

Refer to answer 89, page no. 150 of MTG CBSE Champion Mathematics, Class-11.

29. Total number of students = 25
Total students chosen for an excursion party = 10
Case I : If all 3 students joined
Then, we have to select 7 students out of 22.
$\therefore$ Total number of combinations $= {}^{22}C_7$
Case II : If all the 3 students have not joined
Then, we have to select 10 students out of 22.
Total number of combinations $= {}^{22}C_{10}$
$\therefore$ Total number of ways $= {}^{22}C_7 + {}^{22}C_{10}$
$= 170544 + 646646 = 817190$

30. Given letters are A, G, A, I, N
Total words starting with A are 4! = 24

Total words starting with G are $= \frac{4!}{2!} = 12$

Total words starting with I are $= \frac{4!}{2!} = 12$

49^{th}, words starting with N is NAAGI
50^{th}, words starting with N is NAAIG.

31. Number of boys = 9, Number of girls = 4
Total members in a committee = 7
(i) Number of ways forming a committee having atleast 3 girls $= {}^4C_3 \times {}^9C_4 + {}^4C_4 \times {}^9C_3 = 504 + 84 = 588$
(ii) Number of ways forming a committee having atmost 3 girls
$= {}^4C_3 \times {}^9C_4 + {}^4C_2 \times {}^9C_5 + {}^4C_1 \times {}^9C_6 + {}^4C_0 \times {}^9C_7$
$= 504 + 756 + 336 + 36 = 1632$

OR

Refer to answer 45, page no. 146 of MTG CBSE Champion Mathematics class-11.

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32. Here, ${}^{n}P_{r} = {}^{n}P_{r+1}$

$\Rightarrow \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$

$\Rightarrow \frac{1}{n-r} = 1 \Rightarrow n-r = 1$ ...(i)

Also, ${}^{n}C_{r} = {}^{n}C_{r-1}$

$\Rightarrow \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!(r-1)!}$

$\Rightarrow \frac{1}{(n-r)!\,r(r-1)!} = \frac{1}{(n-r+1)(n-r)!(r-1)!}$

$\Rightarrow \frac{1}{r} = \frac{1}{n-r+1}$

$\Rightarrow n-2r = -1$ ...(ii)

From (i) and (ii), we get, $n = 3$ and $r = 2$

33. (i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \frac{7!}{3!} = 840.$

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three A's are alike.

So, total number of words $= \frac{5!}{3!} = 20.$

(iii) Considering H and N together we have 7–2 + 1 = 6 letters out of which three A's are alike. These six letters can be arranged in $\frac{6!}{3!}$ ways. But H and N can be arranged among themselves in 2! ways.

$\therefore$ Required number of words $= \frac{6!}{3!} \times 2! = 240.$

34. Total number of persons = 3 girls + 9 boys = 12

Total number of numbered seats = 2 × 3 + 4 × 2 = 14

So, total number of ways in which 12 persons can be seated on 14 seats = Number of arrangements of 14 seats by taking 12 at a time = ${}^{14}P_{12}$.

Three girls can be seated together in a back row on adjacent seats in the following way:

So, the total number of ways in which three girls can be seated together in a back row on adjacent seats = 4 × 3!

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_{9}$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}P_{9} \times 4 \times 3!$.

OR

Refer to answer 92, page no. 150 of MTG CBSE Champion Mathematics class-11.

35. A candidate can select 6 questions in the following 6 ways :

(i) 2 from part *A*, 1 from part *B* and 3 from part *C*.
(ii) 2 from part *A*, 2 from part *B* and 2 from part *C*.
(iii) 2 from part *A*, 3 from part *B* and 1 from part *C*.
(iv) 3 from part *A*, 1 from part *B* and 2 from part *C*.
(v) 3 from part *A*, 2 from part *B* and 1 from part *C*.
(vi) 4 from part *A*, 1 from part *B* and 1 from part *C*.

$\therefore$ Required no. of ways $= C(6,2) \cdot C(3,1) \cdot C(3,3) + C(6,2) \cdot C(3,2) \cdot C(3,2) + C(6,2) \cdot C(3,3) \cdot C(3,1) + C(6,3) \cdot C(3,1) \cdot C(3,2) + C(6,3) \cdot C(3,2) \cdot C(3,1) + C(6,4) \cdot C(3,1) \cdot C(3,1)$

$= \frac{6\times5}{2} \times \frac{3}{1} \times 1 + \frac{6\times5}{2} \times \frac{3}{1} \times \frac{3}{1} + \frac{6\times5}{2} \times 1 \times 3 + \frac{6\times5\times4}{6} \times \frac{3}{1} \times \frac{3}{1} + \frac{6\times5\times4}{6} \times \frac{3}{1} \times 1 + \frac{6\times5}{2} \times \frac{3}{1} \times \frac{3}{1}$

$= 45 + 135 + 45 + 180 + 180 + 135 = 720$

OR

Refer to answer 55, page no. 147 of MTG CBSE Champion Mathematics class-11.

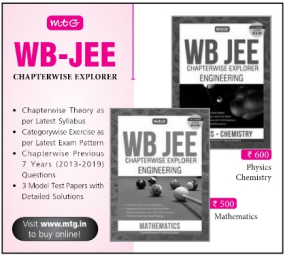
36.

10 Questions

I (5 Questions)	II (5 Questions)
Case (i) 2	4
Case (ii) 3	3
Case (iii) 4	2

$\therefore$ Total no. of ways $= {}^{5}C_{2} \times {}^{5}C_{4} + {}^{5}C_{3} \times {}^{5}C_{3} + {}^{5}C_{4} \times {}^{5}C_{2}$

$= 10 \times 5 + 10 \times 10 + 5 \times 10 = 200$



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MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 4 : Permutations and Combinations, Mathematical Reasoning, Mathematical Induction

Time Taken : 60 Min.

Only One Option Correct Type

1. A person always prefers to eat 'parantha' and 'vegetable dish' in his meal. How many ways can he make his platter in a marriage party, if there are three types of 'paranthas', four types of 'vegetable dish', three types of 'salads' and two types of 'sauces'?
(a) 3360 (b) 4096
(c) 3000 (d) none of these

2. For all positive integers $n > 1$,
$\{x(x^{n-1} - na^{n-1}) + a^n(n-1)\}$ is divisible by
(a) $(x-a)^2$ (b) $x-a$
(c) $2(x-a)$ (d) $x+a$

3. The number of ways in which 10 candidates $A_1, A_2, ..., A_{10}$ can be ranked such that A_1 is always above A_{10} is
(a) 5! (b) 2(5!)
(c) 10! (d) $\frac{1}{2}(10!)$

4. The proposition $(p \to \sim p) \wedge (\sim p \to p)$ is
(a) a tautology
(b) a contradiction
(c) neither a tautology nor a contradiction
(d) a tautology and a contradiction

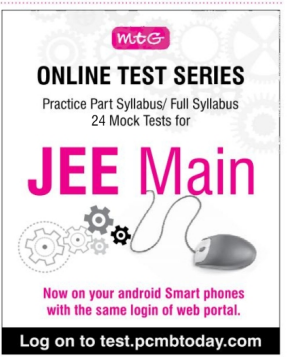
5. The maximum number of points of intersection of 6 circles is
(a) 25 (b) 24 (c) 50 (d) 30

6. If p : a man is happy.
q : a man is rich.
Then, the statement, "If a man is not happy, then he is not rich" is written as
(a) $\sim p \to \sim q$ (b) $\sim q \to p$
(c) $\sim q \to \sim p$ (d) $q \to \sim p$

One or More Than One Option(s) Correct Type

7. If p, q and r are simple propositions, then $(\sim p \vee q) \Rightarrow r$ is true, when p, q and r are, respectively
(a) T, F, T (b) T, T, T
(c) F, T, T (d) F, F, F

8. A fair coin is tossed n times. Let a_n denote the number of cases in which no two heads occur consecutively. Then which of the following is true?
(a) $a_1 = 2$ (b) $a_2 = 3$
(c) $a_5 = 13$ (d) $a_6 = 55$



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9. Two players P_1 and P_2 plays a series of $2n$ games. Each game can result in either a win or loss for P_1. Total number of ways in which P_1 can win the series of these games, is equal to

(a) $\frac{1}{2}(2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2}(2^{2n} - 2 \cdot {}^{2n}C_n)$

(c) $\frac{1}{2}\left(2^{2n} - \frac{2n!}{n!\,n!}\right)$ (d) none of these

10. For all positive integers n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is a/an

(a) integer (b) rational number
(c) irrational number (d) none of these

11. The proposition $\sim (p \Rightarrow q) \Rightarrow (\sim p \vee \sim q)$ is

(a) a tautology (b) a contradiction
(c) either (a) or (b) (d) neither (a) nor (b)

12. All the five-digit numbers in which each successive digit exceeds its predecessor are arranged in the increasing order. The 105^{th} number does not contain the digit

(a) 1 (b) 3 (c) 4 (d) 5

13. If p and q are simple propositions, then $(\sim p \wedge q) \vee (\sim q \wedge p)$ is false, when p and q are respectively

(a) T, T (b) F, T
(c) F, F (d) none of these

Comprehension Type

Two numbers x and y are drawn without replacement from the set of the first 15 natural numbers. The number of ways of drawing the numbers such that

14. $x^3 + y^3$ is divisible by 3, is

(a) 21 (b) 33 (c) 35 (d) 69

15. $x^2 - y^2$ is divisible by 3, is

(a) 21 (b) 33 (c) 35 (d) 55

Matrix Match Type

16. Match the following :

Column-I		Column-II	
P.	The greatest positive integer, which divides $(n+2)(n+3)(n+4)(n+5)(n+6)$ for all $n \in N$, is	1.	120
Q.	A man moves one unit distance for each step he takes either along the positive x-direction or negative x-direction. The number of ways he moves from the point (0, 0) to (2, 0) in at most 10 steps is	2.	3720
R.	Given 5 different green dyes, 4 different blue dyes and 3 different red dyes, the number of combinations of dyes taking atleast one green dye and one blue dye, is	3.	286
S.	Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to	4.	7200

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	1	3
(c)	4	2	3	1
(d)	1	3	2	4

MATHEMATICS KING

Integer Answer Type

17. For each natural number, the statement $P(n) = 2^{3n} - 1$ is divisible by _____ .

18. The number of ordered pairs of natural numbers (a, b) such that $\frac{ab}{a+b} = 511$ is _____ .

19. The number of ways in which we can choose 5 letters from the word 'INTERNATIONAL' is N. Then $\frac{N}{128}$ is equal to _____ .

20. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then value of n is _____ .

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No. of questions attempted .....
No. of questions correct .....
Marks scored in percentage .....

Check your score! If your score is

> 90%	EXCELLENT WORK ! You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK ! You can score good in the final exam.
74-60%	SATISFACTORY ! You need to score more next time.
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Master the Concepts

OF PERMUTATIONS AND COMBINATIONS

Class XI

DEFINITION OF FACTORIAL

The product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor n$.
That is $n! = n(n-1)(n-2) \ldots\ldots 3 \times 2 \times 1$
Note : Factorials of proper fractions and of negative integers are not defined. Factorial n is defined only for whole numbers and we defined 0! as 1, *i.e.*, $0! = 1$.

Properties of Factorial

(a) $n! = 1 \times 2 \times \ldots\ldots \times (n-1) \times n = [1 \times 2 \times \ldots\ldots \times (n-1)] \, n = (n-1)!n$
Thus, $n! = n(n-1)!$

(b) If two factorials, *i.e.*, $x!$ and $y!$ are equal, then $(x, y) = (0, 1)$ or $(1, 0)$ or $(k, k) \; \forall \; k \in N_0 = N \cup \{0\}$

(c) $n!$ ends in 0, for all $n > 4$, (Number of 5's in $n!$, $n > 4$, is always less than the number of 2's. Therefore, for every 5, there is a 2. Hence, $n!$, $n > 4$, ends in 0).

FUNDAMENTAL PRINCIPLE OF COUNTING

Addition Principle

If there are
n_1 ways for the event E_1 to occur
n_2 ways for the event E_2 to occur
...
...
...
n_k ways for the event E_k, to occur
where $k \geq 1$ and if these are pair-wise disjoint (or mutually exclusive), then the number of ways for at least one of the events $E_1, E_2, \ldots\ldots, E_k$ to occur in $n_1 + n_2 + \ldots\ldots + n_k = \sum_{n=1}^{k} n_i$.

Multiplication Principle

If an event E can be decomposed into r ordered sub events $E_1, E_2, \ldots\ldots, E_r$ and if there are n_1 ways (independent to other sub events) for E_1 to occur, n_2 ways (independent to other sub events) for the event E_2 to occur, ... , n_r ways (independent to other sub events) for E_r to occur, then the total number of ways for the event E to occur is given by $n_1 \times n_2 \times \ldots\ldots \times n_r$.

DEFINITION OF COMBINATION

Let A, B, C, D be four distinct objects. The number of ways in which we can select two objects out of A, B, C and D is six and these are AB, AC, AD, BC, BD and CD.
These ways of selection of two objects from four different objects are also known as combinations of A, B, C and D taken two at a time or we can say grouping of A, B, C and D taken two at a time or we can say grouping of A, B, C and D taken two at a time.
A combination of objects is merely a selection (suppress order) from a given lot of objects.
The number of selections of r objects at a time out of n distinct, is $\frac{n!}{r!(n-r)!}$.
This number is denoted as ${}^nC_r$ or $C(n, r)$ or $\binom{n}{r}$.
In general

$$\binom{n}{r} = \begin{cases} \frac{n!}{r!(n-r)!}, & 0 \leq r \leq n;\ r, n \in N_0 \\ 0, & \text{for } r < 0 \text{ or } r > n;\ n \in N_0,\ r \in Z \end{cases}$$

Note : $\binom{0}{0}$ is defined as 1.

Properties of $\binom{n}{r}$; $0 \leq r \leq n$; $r, n \in N_0$

(i) $\binom{n}{0} = \binom{n}{n} = 1$

(ii) $\binom{n}{r} = \binom{n}{n-r}$

(iii) If $\binom{n}{r} = \binom{n}{k}$, then $r = k$ or $n - r = k$

(iv) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

(v) $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$ or $r\binom{n}{r} = n\binom{n-1}{r-1}$

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(vi) $\frac{1}{r+1}\binom{n}{r}=\frac{1}{n+1}\binom{n+1}{r+1}$

(vii) $\binom{n}{r}=\frac{n-r+1}{r}\binom{n}{r-1}$ or $\frac{\binom{n}{r}}{\binom{n}{r-1}}=\frac{n-r+1}{r}$

(viii) $\binom{n}{r}=\frac{n}{n-r}\binom{n-1}{r}$

(ix) (a) If n is even, $\binom{n}{r}$ is greatest for $r=\frac{n}{2}$.

(b) If n is odd, $\binom{n}{r}$ is greatest for $r=\frac{n-1}{2}, \frac{n+1}{2}$.

In general $\binom{n}{r}$ is maximum at $r=\left\lfloor\frac{n}{2}\right\rfloor, \left\lceil\frac{n}{2}\right\rceil$

Some Applications of Combinations

- The number of ways to select r objects from n distinct objects where p particular objects should always be included in the selection = ${}^{n-p}C_{r-p}$.
- The number of ways to select r objects from n different objects where p particular objects should never be included in the selection ${}^{n-p}C_r$.
  Note : There are problems in which constraints are to select minimum (atleast) or maximum (atmost) number of objects in the selection. In these problems, we should always make cases to select objects. If we do not make cases, we will get wrong answer.
- The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects = $2^n - 1$.
- The number of ways to select 0 or more objects from n distinct objects = 2^n.
- The number of ways to select at least r objects from n distinct objects = $2^n - 1 - {}^nC_1 - {}^nC_2 - {}^nC_3 - .... - {}^nC_{r-1}$ or ${}^nC_r + {}^nC_{r+1} + {}^nC_{r+2} + ... + {}^nC_n$.
- The number of ways to select one or more objects (or at least one object) from n identical object = n.
- The number of ways to select r objects from n identical objects is 1.
- The number of ways to select 0 or more objects from n identical objects = $n + 1$.
- The number of ways to select atleast r objects from n identical objects is $n - (r - 1) = n - r + 1$.
- The total number of selections of some or all out of $(p + q + r)$ objects where p are alike of one kine, q are alike of second kind and rest r are alike of third kind is $(p + 1)(q + 1)(r + 1) - 1$. [Using fundamental principle of counting].
- The number of ways to select 0 or more objects from p alike of one kind, q alike of second kind, r alike of third kind and n distinct objects = $(p + 1)(q + 1)(r + 1)2^n$.
- The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object each of first, second and third kind and atleast one object from n different kind = $pqr(2^n - 1)$.
- The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n distinct objects such that selection includes at least one object of each kind = $pqr2^n$.

Geometrical Applications

- Number of point of intersection of 'n' non-concurrent and non-parallel lines is ${}^nC_2$.
- Number of lines that can be drawn, passing though any 2 points out of n given points in which no three of them are collinear, is ${}^nC_2$.
- Number of triangles that can be formed, by joining any three points out of n given points in which no three of them are collinear is ${}^nC_3$.
- Number of diagonals that can be drawn in a 'n' sided polygon is $\frac{n(n-3)}{2}$.

Combination with Repetitions Allowed

Here we will discuss combinations of n distinct types of objects (each type of object is available in sufficient quantity) taken r at a time when each object can be repeated any number of times in a combination.

Suppose three distinct types of objects A, B, C are given. We have to select two objects from A, B, C and in our selection we can include A, B, C repeatedly any number of times. This selection can be done infollowing ways.

AA, BB, CC, AB, AC, BC, *i.e.*, 6 ways.

This number 6 cannot be obtained using formula ${}^nC_r$ as here repetition of objects in allowed. To find answer to this type of problem, where repetition of objects is allowed, we use the following formula : Number of ways to select r objects from n distinct

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objects where each object can be selected any number of times is ${}^{n}H_{r}$.

$${}^{n}H_{r} = \binom{n+r-1}{r}$$

PERMUTATIONS (ARRANGEMENTS)

A permutation of given objects is an arrangement of the objects in a line or row, unless specified otherwise. These arrangements can be generated by changing the relative positions of objects in the row. Every possible relative order between the objects is taken into account.

Number of Permutations (Arrangements, Orders) of *n* Distinct Objects taken all at a Time

- The total number of permutations of n distinct objects = $n!$
- The number of ways to arrange n distinct objects such that p particular objects remain together in the arrangement $(n - p + 1)!\, p!$
- The number of ways to arrange n different objects such that p particular objects are always separated $= {}^{n-p+1}C_{p} \times (n - p)! \times p!$

Number of Permutations (Arrangements, Orders) of *n* Distinct Objects taken *r* at a Time

- The total numbers of permutations of r objects, out of n distinct objects, is $\dfrac{n!}{(n-r)!}$, $0 \le r \le n$.

  This number is denoted as ${}^{n}P_{r}$ or $P(n, r)$.
- The number of ways to select and arrange (permutate) r objects from n distinct objects such that arrangement should always include p particular objects $= {}^{n-p}C_{r-p} \times r!$.
- The number of ways to select and arrange r objects from n distinct objects such that p particular objects are always excluded in the selection $= {}^{n-p}C_{r} \times r!$.

Permutation of Objects when not all Objects are Distinct

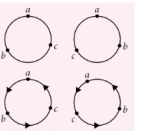
- Let there be n_1 A_1's, n_2 A_2's, ..., n_k A_k's. Then the number of permutations $= \dfrac{(n_1 + n_2 + ... + n_k)!}{n_1!\, n_2! .... n_k!}$

  (This number is known as a multinomial coefficient).

Arrangement of *n* Distinct Objects with Repetition of Objects

- Total number of ways to permutate n distinct types of objects (each type of object available in sufficient quantity) taken r at a time when objects can be repeated any number of times is n^r.

Introduction to Circular Permutation



When objects are to be arranged (ordered) in a circle instead of a row, it is known as circular permutation. For example, three objects a, b, c can be permutated in a circle as shown in figure.

Number of ways to arrange a, b, c in circle is not same as number of ways to arrange a, b, c in a row. This is because arrangements abc, bca, cab in a row are same in circle as shown in the figure.

Similarly arrangements acb, cba, bac in a row are same in circle as shown in the figure.

Some Important Results

- The number of circular permutations of n distinct objects is $(n - 1)!$
- Number of circular permutations of n distinct objects such that clockwise and anti-clockwise arrangements of objects are same $= \dfrac{(n-1)!}{2}$, $n \ge 3$.
- Number of circular permutations of 'n' distinct things taken 'r' at a time $= \dbinom{n}{r}(r-1)!$ (when clockwise and anti-clockwise orders are taken as different).
- If clockwise and anti-clockwise orders are taken as same, then the required number of circular permutations $= \dfrac{\dbinom{n}{r}(r-1)!}{2}$, $r \ge 3$.

Division and Distribution of Non-Identical Items in Fixed Size

1. Number of ways in which $(m + n + p)$ distinct objects can be divided into 3 unequal (groups contain unequal number of objects) unnumbered groups containing m, n, p objects $= {}^{m+n+p}C_{m} \; {}^{n+p}C_{n} \; {}^{p}C_{p} = \dfrac{(m+n+p)!}{m!n!p!}$

   (Here among m, n, p no two are equal)
2. Number of ways in which $(m + n + p)$ distinct objects can be divided and distribute into 3 unequal numbered groups (Here among m, n, p no two are equal containing m, n, p objects)

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= Number of ways to divide $(m + n + p)$ objects in 3 groups × Number of ways to distribute 'division-ways' among groups

= Number of ways to divide $(m + n + p)$ objects in 3 groups × (Number of groups)! $\frac{(m+n+p)!}{m!n!p!} \times 3!$

Note : Above formulae are written for dividing objects into 3 groups but in case group are more, then also we follow the same approach.

3. Number of ways to divide (mn) distinct objects equally in m unnumbered group (each group get n objects)

$$\binom{mn}{n}\binom{mn-n}{n}\binom{mn-2n}{n}\cdots\binom{n}{n}\frac{1}{m!}=\frac{(mn)!}{(n!)^m m!}$$

4. Number of ways to divide (mn) distinct objects equally in m numbered group (each group gets n objects)

$$=\frac{(mn)!}{(n!)^m m!}\times m!=\frac{mn!}{(n!)^m}$$

5. Number of ways to divide $(ma + nb + pc)$ distinct objects (out of a, b, c no two numbers are equal) in $(m + n + p)$ unnumbered groups such that m groups contains a objects each, n group contains b objects each, p group contains c objects each

$$=\frac{(ma+nb+pc)!}{(a!)^m(b!)^n(c!)^p m!n!p!}$$

6. Number of ways to divide and distribute $(ma + nb + pc)$ distinct objects (out of a, b, c no two numbers are equal) in $(m + n + p)$ numbered group such that m groups contains a objects each, n groups contains b objects each, p groups contains c object each

$$=\frac{(ma+nb+pc)!}{(a!)^m(b!)^n(c!)^p m!n!p!}\times(m+n+p)!$$

NUMBER OF INTEGRAL SOLUTIONS

- The number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots\ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$
- The number of non-negative integral solutions of the inequality $x_1 + x_2 + \ldots\ldots + x_r \le n$ is ${}^{n+r}C_r$. This is done by introducing a non-negative dummy variable x_{r+1} on the left hand side of inequality. The situation will be reduced to finding the number of non-negative integral solutions of the equation $x_1 + x_2 + \ldots\ldots + x_r + x_{r+1} = n$ which will be ${}^{n+(r+1)-1}C_{(r+1)-1}$ *i.e.*, ${}^{n+r}C_r$
- The number of integral solutions of the equation $x_1 + x_2 + \ldots\ldots + x_r = n$ subjected to condition $a_1 \le x_1 \le b_1$, $a_2 \le x_2 \le b_2$, ..., $a_r \le x_r \le b_r$ is equal to the coefficient of x^n in the expansion of

$$(x^{a_1}+x^{a_1+1}+\ldots\ldots+x^{b_1})(x^{a_2}+x^{a_2+1}+\ldots+x^{b_2})\ldots\ldots(x^{a_r}+x^{a_r+1}+\ldots+x^{b_r})$$

- Consider $x_1 + x_2 + x_3 + \ldots\ldots x_r = n$ ...(i)
where $x_1 \ge a_1$, $x_2 \ge a_2$, ... , $x_r \ge a_r$, all a_i's are integers.
Take $x_1 = a_1 + x_1'$
$x_2 = a_2 + x_2'$, etc.,
where $x_1' \ge 0$, $x_2' \ge 0$, ..., $x_r' \ge 0$
Eq. (i) reduces to
$(a_1 + a_1 + \ldots\ldots + a_r) + x_1' + x_2' + \ldots\ldots + x_r' = n$
$\Leftrightarrow \quad x_1' + x_2' + \ldots + x_r' = n - (a_1 + a_2 + \ldots + a_r)$ ...(ii)
For every solution of eq. (i), we can write a corresponding solution of eq. (ii) and for every solution of eq. (ii), we can write a corresponding solution of eq. (i).
Number of solutions of eq. (i),
= Number of non-negative integral solutions of eq. (ii)

$$=\frac{(n+r-1-(a_1+a_2+\ldots+a_r))!}{(r-1)!(n-(a_1+a_2+\ldots+a_r))!}$$

DERANGEMENT

A derangement of 1, 2, ..., n is a permutation of the numbers such that no number occupies its natural position. Thus, (2, 3, 1, ...) and (3, 1, 4, 2, ...) are derangements. On the other hand, (2, 4, 3, 5, 1, ...) is not a derangements as 3 is at the 3rd position.

The total number of derangements of 1, 2, ..., n will be denoted by D_n.

It is easy to realise that $D_1 = 0$, $D_2 = 1$ and $D_3 = 2$, etc.

$$D_n = n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{(-1)^n}{n!}\right]$$

DIRICHLET'S (OR PIGEON HOLE) PRINCIPLE (PHP)

Let $k, n \in \mathbb{N}$. If at least $kn + 1$ objects are distributed among k boxes, then atleast one of the box, must contain atleast $(n + 1)$ objects. It particular, if atleast $(n + 1)$ objects are put into n boxes, then atleast one of the box must contain atleast two objects. For arbitrary n objects and m boxes this generalizes to atleast one box will contain atleast

$$\left[\frac{n-1}{m}\right]+1 \text{ objects.}$$

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PROBLEMS

One or More than One Option(s) Correct Type

1. Let $S = \{1, 2, 3, 4\ldots\ldots, (n+1)\}$, where $n \geq 6$ and let $T = \{(x, y, z) \mid x, y, z \in S, x < z, y < z\}$, then number of members of T is equal to

(a) ${}^{n+1}C_2 + {}^{n+1}C_3$ (b) ${}^{n+1}C_2 + 2\ {}^{n+1}C_3$

(c) $\sum_{k=1}^{n} k^2$ (d) $\sum_{k=1}^{n-1} k^2$

2. The number of ordered triplets (x, y, z) of non negative integers satisfying the conditions.

$x + y + z \leq 100$ and $x \leq y \leq z$

(a) if x is odd is 14724 (b) if x is odd is 14722

(c) if x is even is 16065 (d) if x is even is 16164

3. Let 5 letter words are formed using the letters of the word 'CALCULUS'. Then

(a) The number of all such possible words is 1110.

(b) The number of words with exactly one alike pair of letters is 720.

(c) The number of words with exactly two alike pair of letters is 270.

(d) The number of words with exactly one alike pair of letters is 480.

4. The integers from 1 to 100 are written in order around a circle. Starting at 1, every fifteenth number is marked *(i.e.,* 1, 16, 31, ...). This process is continued until a number is reached which has already been marked, then marked numbers are

(a) 11 (b) 17 (c) 986 (d) 996

5. The number of ordered quadruples (a_1, a_2, a_3, a_4) of positive odd integers that satisfy $a_1 + a_2 + a_3 + a_4 = 32$ is equal to

(a) ${}^{31}C_3$ (b) ${}^{17}C_3$

(c) 4495 (d) $\frac{9}{2}({}^{17}C_3) - {}^{17}C_4$

6. Two class rooms A and B have capacity of 25 and $n - 25$ seats respectively. A_n denotes the number of possible seating arrangements of persons in room A when n persons are to be seated in these rooms, starting from room A which is to be filled up to its capacity. If $A_n - A_{n-1} = 25! ({}^{49}C_{25})$, then n is divisible by

(a) 5 (b) 2 (c) 4 (d) 6

7. Consider a square of dots, coloured red or blue with 20 rows and 20 columns. Whenever two dots of same colour are adjacent in the same row or column, they are joined by a segment of their common colour. Adjacent dots of unlike colours are joined by a black segment. There are 219 red dots, 39 of them on the border of the array, not at the corners. If there are 237 black segments, then the number of red and blue segments are n_1 and n_2 respectively, then

(a) $n_1 = 300$ (b) $n_1 = 310$

(c) $n_2 = 223$ (d) $n_2 = 213$

8. Let N be the number of ways of filling three boxes A, B and C by 12 or less number of identical balls, such that no box is empty, box B has at least 3 balls and box C has at most 5 balls. Then

(a) N is divisible by 2 but not by 5

(b) N is divisible by 5 but not by 11

(c) Sum of the digits of number N is 2

(d) Sum of the digits of number N is 4

Comprehension Type

Paragraph for Question No. 9 to 11

If a set A has n elements, then the number of subsets of A containing exactly r elements is ${}^{n}C_r$ and the number of all subsets of A is 2^n. Now answer the following questions. If a set A has 7 elements and a subset P of A is selected. After noting the elements they are placed back in A. Again subset Q is selected. Then the number of ways of selecting P and Q such that

9. P, Q have no common element is

(a) 2835 (b) 128 (c) 3432 (d) 2187

10. P and Q have exactly 3 elements in common is

(a) 2835 (b) 128 (c) 3432 (d) 2187

11. P and Q have equal number of elements is (P and Q may be null sets)

(a) 2835 (b) 128 (c) 3432 (d) 2187

Paragraph for Question No. 12 to 13

Each of 5 woman who attend a banquet checks her coat and hat with the receptionist on arrival. Upon leaving, each woman is given a coat and a hat at random.

12. The number of ways these coats and hats may be distributed such that nobody gets back both her coat and her hat is

(a) 2048 (b) 1936

(c) 1956 (d) 1978

13. The number of ways these coats and hats may be distributed such that nobody gets back either her coat or her hat is

(a) 14400 (b) 118000

(c) 11842 (d) 11844

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Paragraph for Question No. 14 to 15

A bag has 5 white marbles and 5 blue marbles, marbles are drawn one by one and without replacement till all marbles are drawn. Marbles of each colour are indistinguishable. Then

14. In how many ways may one draw the marbles out of the bag if all blue are drawn in odd numbered draws, or all blue are drawn in even numbered draw?

(a) 1202 (b) 1448
(c) 1512 (d) 1668

15. In how many ways will all red marbles come before any of the blue marbles?

(a) 1496 (b) 1287
(c) 1024 (d) 967

Numerical Answer Type

16. A game uses a deck of n different cards with $n \geq 6$. If the number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn, then the sum of the digits of n is _____ .

17. Put numbers 1, 2, 3, 4, 5, 6, 7 and 8 at the vertices of a cube, such that the sum of any three numbers on any face is not less than 10. The minimum sum of the four number on a face is k, the $k/2$ is equal to _____ .

18. Number of natural numbers n for which $n!$ ends with precisely 25 zeroes is _____ .

19. Let N be the number of ordered 6-tuples $(x_1, x_2, x_3, x_4, x_5, x_6)$ of positive integers satisfying $x_1 + x_2 + x_3 + 3x_4 + 3x_5 + 5x_6 = 21$. Then $\sqrt{N}$ is equal to _____ .

20. Let N is the number of 4 element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \ldots\ldots, 20\}$ such that $a + b + c + d$ is divisible by 3. Then the sum of the digits of N is equal to _____ .

21. Suppose A_1, A_2, ..., A_{20} is a 20 sided regular polygon. Let N be the number of non isosceles triangles that can be formed whose vertices are the vertices of the polygon but whose sides are not the sides of the polygon. Then $\sqrt{\frac{N}{10}}$ is equal to_____ .

22. Let N be the number of 6 digit numbers such that the digits of each number are all from the set {1, 2, 3, 4, 5} and any digit that appears in the number appears atleast twice. Then the last digit of N is _____ .

SOLUTIONS

1. (b, c) : $z = 2$, number of $(x, y) = 1 \times 1$
$z = 3$, number of $(x, y) = 2 \times 2$
$z = 4$, number of $(x, y) = 3 \times 3$
$z = n + 1$, number of $(x, y) = n \times n$
Alternate : $x < y < z$ or $y < x < z$
$\Rightarrow$ Number of $(x, y, z) = 2 \times {}^{n+1}C_3$
or $x = y < z$
$\Rightarrow$ Number of $(x, y, z) = {}^{n+1}C_2$
Total number of $(x, y, z) = {}^{n+1}C_2 + 2\ {}^{n+1}C_3$

2. (b, c) : If x is odd
$x = 2k + 1$,
$y = 2k + 1$, $z = 2k + 1, 2k + 2, \ldots\ldots, 98 - 4k \rightarrow (98 - 6k)$ ways
$y = 2k + 2$, $z = 2k + 2, 2k + 3, \ldots\ldots, 97 - 4k \rightarrow (96 - 6k)$ ways
$\vdots$
$y = 49 - k$, $z = 49 - k, 50 - k \rightarrow 2$ ways

$\therefore$ Number of triplets $= \sum_{k=0}^{16} (98 - 6k) + (96 - 6k) + \ldots + 2$

$= \sum_{k=0}^{16} (49 - 3k)(50 - 3k) = 14722$

If x is even
$x = 2k$, $y = 2k$, $z = 2k, 2k + 1, \ldots,100 - 4k \rightarrow (101 - 6k)$ ways
$y = 2k + 1$, $z = 2k + 1, \ldots\ldots, 99 - 4k \rightarrow (99 - 6k)$ ways
$\vdots$
$y = 50 - k$, $z = 50 - k \rightarrow 1$ way

$\therefore$ Number of triplets $= \sum_{k=0}^{16} ((101 - 6k)(99 - 6k) + \ldots + 1)$

$= \sum_{k=0}^{16} (51 - 3k)^2 = 16065$

3. (a, b, c) : (a) (2A, 3D) + (2A, 2A, 1D) + (5D)

$= {}^3C_1\ {}^4C_3\ \frac{5!}{2!} + {}^3C_2\ {}^3C_1\ \frac{5!}{2!\,2!} + 5! = 1110$

(b) 2A, 3D $= {}^3C_1\ {}^4C_3\ \frac{5!}{2!} = 720$

(c) 2A, 2A, 1D $= {}^3C_2\ {}^3C_1\ \frac{5!}{2!2!} = 270$

4. (a, c, d) : In 1st round, number are of form $15k + 1$, so last number is 991
In 2nd round, first number is $991 + 15 - 1000 = 6$, so numbers are of the form $15k + 6$ and so last number is 996.

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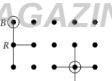
In 3rd round, first number is 996 + 15 − 1000 = 11, so number are of the form $15k + 11$ and so last number is 986.
In 4th round, first number is 986 + 15 − 1000 = 1

5. (b, d) : $(2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) = 32$
$\Rightarrow k_1 + k_2 + k_3 + k_4 = 14$
$\Rightarrow {}^{14+4-1}C_{4-1} = {}^{17}C_3 = 680$

6. (a, b) : $A_n - A_{n-1} = 25!\ ({}^{49}C_{25})$
$\Rightarrow {}^{n}C_{25}(25!) - {}^{n-1}C_{25}(25!) = 25!\ ({}^{49}C_{25})$
$\Rightarrow {}^{n}C_{25} - {}^{n-1}C_{25} = {}^{49}C_{25}$
$\Rightarrow {}^{n-1}C_{24} = {}^{49}C_{24} \Rightarrow n - 1 = 49$
$\Rightarrow n = 50$

7. (a, c) : In each row, there are 19 segments (Since there are 20 points in each row.) There are 20 rows and hence there are $20 \times 19 = 380$ horizontal segments.
Similarly, there are $20 \times 19 = 380$ vertical segments (Since, there are 20 columns with 19 segments in each column).
Therefore, the total number of segments = 760.
Number of black segments = 237.
Number of segments which are either blue or red = 523.



Let r denote the number of red segments and each red segment has 2 red points as the end point of the segment and each black segment has one end point coloured blue and the other end point coloured red.
So, the total number of times a red dot becomes an end point of a segment is
$= 2 \times r + 237 = 2r + 237$ ...(i)
There are altogether 219 red dots and of these, 39 are on the border.
So, the number of red dots in the interior is 180.
Each red dot on the border accounts for 3 segments (Since none of the red dots is on the corner).
So, the number of segments for which each red point on the border becomes the end points 3.
So, the total number of segments to which the 39 border red dots are end points $39 \times 3 = 117$.
Each of the 180 red points in the interior becomes the end point for 4 segments.
So, the total number of segments for which the 180 red points are the end points $= 180 \times 4 = 720$.
So the total number of times a red dot becomes an end point, *i.e.*, total number of red ends
$= 117 + 720 = 837$ ...(ii)

Hence, Eqs. (i) and (ii) represent the same number, the result,
$\therefore \quad 2r + 237 = 837$
$\therefore \quad r = 300.$
i.e., the number of red segments = 300
and the number of blue segments = 523 − 300 = 223.

8. (c) : Suppose box A has z_1 balls, box B has x_2 balls and box C has x_3 balls. Then,
$x_1 + x_2 + x_3 \le 12,\ x_1 \ge 1,\ x_2 \ge 3,\ 1 \le x_3 \le 5$
Let $x_4 = 12 - (x_1 + x_2 + x_3)$. Then
$x_1 + x_2 + x_3 + x_4 = 12$
$(1 \le x_1 \le 8,\ 3 \le x_2 \le 10,\ 1 \le x_3 \le 5$ and $0 \le x_4 \le 7)$
The required number = Coefficient of x^{12} in
$(x^1 + x^2 + ... + x^8)\ (x^3 + x^4 + ... + x^{10})\ (x^1 + x^2 + ... + x^5)\ (x^0 + x^1 + ... + x^7)$
= Coefficient of x^{12} in $(x + x^2 + x^3 + ...)\ (x^3 + x^4 + x^5 + ...)\ (x + x^2 + ... + x^5)\ (1 + x + x^2 + ...)$
= Coefficient of x^7 in $(1 + x + x^2 + ...)\ (1 + x + x^2 + ...)\ (1 + x + x^2 + x^3 + x^4)\ (1 + x + x^2 + ...)$
= Coefficient of x^7 in $(1 - x)^{-4}\ (1 - x^5)$
= Coefficient of x^7 in $(1 - x^5)\ (1 + {}^4C_1x + {}^5C_2x^2 + {}^6C_3x^3 + ...)$
$= {}^{10}C_7 - {}^5C_2 = 110.$

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9. (d) : Each of the 7 elements have 3 choice. They are $x \in P$ and $x \notin Q$
or $x \notin P$ and $x \in Q$
or $x \notin P$ and $x \notin Q$
For each element there are three of above choices
Hence the number of ways of selecting P or $Q = 3^7 = 2187$

10. (a) : As in both sets three elements are common so, three elements can be chosen in ${}^7C_3$ ways, and rest of the elements can be chosen in any of the above three choices in 3^4 ways.
So, total number of ways $= {}^7C_3 \times 3^4 = 2835$

11. (c) : Let P and Q have same number of elements, say $r (0 \le r \le 7)$
$\Rightarrow$ Total number of ways in which P and Q have same number of elements
$$= \sum_{r=0}^{7} {}^7C_r \cdot {}^7C_r = ({}^7C_0)^2 + ({}^7C_1)^2 + ({}^7C_2)^2 + ... + ({}^7C_7)^2 = 3432$$

12. (b) : The number of ways of distributing coats such that nobody gets back her coats $= 5! - ({}^5C_1 4! - {}^5C_2 3! + {}^5C_3 2! - {}^5C_4 1! + {}^5C_5) = 44$
Number of ways $= 44 \times 44 = 1936$

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13. (d) : Number of ways = $(5!)^2 - {}^5C_1(4!)^2 - {}^5C_2(3!)^2 + {}^5C_3(2!)^2 - {}^5C_4(1!)^2 + {}^5C_5) = 11844$

14. (c) : ${}^7C_5 \frac{8!}{5!3!} + {}^6C_5 \frac{8!}{5!3!} = 1512$

15. (b) : $x_1Rx_2Rx_3Rx_4Rx_5Rx_6Rx_7Rx_8Rx_9$
$x_1 + x_2 + ... + x_9 = 5$
Number of ways $= {}^{5+8}C_8 = 1287$

16. (4) : ${}^nC_6 = 6 \cdot {}^nC_3$
$\Rightarrow (n-3)(n-4)(n-5) = 10 \times 9 \times 8$
$\Rightarrow n = 13$

17. (8) : Suppose that the four numbers on face of the cube is a_1, a_2, a_3, a_4 such that their sum reaches the minimum and $a_1 < a_2 < a_3 < a_4$.

Since the maximum sum of any three numbers less than 5 is 9, we have $a_4 \geq 6$ and $a_1 + a_2 + a_3 + a_4 \geq 16$.
As seen in figure, we have $2 + 3 + 5 + 6 = 16$ and that means minimum sum of four numbers on a face is 16.

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18. (5) : $10 = 2 \times 5$ and $n! = 1 \cdot 2 \cdot 3 ..... n$
Each multiple of 5 contributes one zero
Each multiple of 25 contributes one more zero, etc.
$\therefore$ 100! has 24 zeroes at the end
$\therefore$ $n = 105, 106, ..., 109$

19. (a) : $x_1 + x_2 + x_3 = x, \quad x_4 + x_5 = y \quad z = x_6$
$x + 3y = 21 - 5x_6$
$\Rightarrow x_6 = 1, 2$
$\Rightarrow x + 3y = 16, 11$
$(x, y, x_6) = (4, 4, 1), (7, 3, 1), (10, 2, 1), (5, 2, 2)$
Number of 6-tuples $= {}^{4-3+2}C_2 {}^{4-2+1}C_1 + {}^{7-3+2}C_2 {}^{3-2+1}C_1 + {}^{10-3+2}C_2 {}^{2-2+1}C_1 + {}^{5-3+1}C_1 {}^{2-2+1}C_1$
$= 9 + 30 + 36 + 6 = 81$

20. (a) : Number of ways
$= {}^6C_4 + {}^6C_2 {}^7C_1 {}^7C_1 + {}^6C_1({}^7C_3 + {}^7C_3) + {}^7C_2 {}^7C_2$

All of form $3k$	Two of form $3k$, one of form $3k+1$ and one of form $3k+2$	One of form $3k+1$ or $3k+2$	Two of form $3k+1$ and two of form $3k+2$

$= 15 + 735 + 420 + 441 = 1611$

21. (8)

22. (5) : 6 Alike $\rightarrow 5$

4 Alike + 2 other alike $\rightarrow 5 \times 4 \times \frac{6!}{4!2!} = 300$

3 Alike + 3 other alike $\rightarrow {}^5C_2 \times \frac{6!}{3!3!} = 200$

2 Alike + 2 other alike
+ 2 other alike $\rightarrow {}^5C_3 \times \frac{6!}{2!2!2!} = 900$

N = 900 + 200 + 300 + 5 = 1405

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CONCEPT BOOSTERS

Functions

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

*ALOK KUMAR

DEFINITION

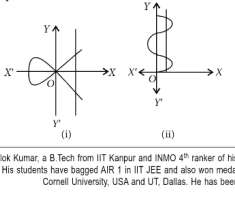
If A and B are two non-empty sets, then a rule f which associates each $x \in A$, to a unique number $y \in B$, is called a function from A to B and we write, $f : A \to B$. We say that 'y' is the image of 'x' under f (or x is the pre-image of y).

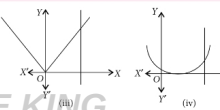
Note : 1. A mapping $f : X \to Y$ is said to be a function if each element in the set X has its image in set Y. It is also possible that there are few elements in set Y which are not the images of any element in set X.

2. Functions can not be multi-valued *i.e.*, every element in set X should have one and only one image in Y.

Testing for a Function by Vertical Line Test

If it is possible to draw a vertical line means line parallel to Y-axis which cuts the graph of given curve at more than one point, then the given relation is not a function and when this vertical line cuts the curve at only one point then it is a function. Figure (iii) and (iv) represents a function.





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NUMBER OF FUNCTIONS

Let X and Y be two finite sets having m and n elements respectively. Then each element of set X can be associated to any one of n elements of set Y. So, total number of functions from set X to set Y is n^m.

DOMAIN, CO-DOMAIN AND RANGE OF FUNCTION

If a function $f : A \to B$ is defined from a set A to set B, then set A is called the domain of function f and set B is called the co-domain of function f. The set of all f-images of the elements of A is called the range of function f.

Methods for finding Domain and Range of Function

- **Domain**

(i) Expression under even root (*i.e.*, square root, fourth root etc.) ≥ 0.

(ii) Denominator $\neq 0$.

(iii) Let domain of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively.

(a) The domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$.

(b) The domain of $\dfrac{f(x)}{g(x)}$ is $D_1 \cap D_2 - \{x : g(x) = 0\}$.

(c) Domain of $\left(\sqrt{f(x)}\right) = D_1 \cap \{x : f(x) \geq 0\}$.

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- **Range**

(i) If domain is the set of finite number of points, then the range is set of corresponding values of $f(x)$.

(ii) If domain is R or $R - [$some finite points$]$, then express x in terms of y and find the values of y for which x exists.

(iii) If domain is a finite interval, then find the least and greatest value for range using monotonicity.

ALGEBRA OF FUNCTIONS

Let $f(x)$ and $g(x)$ be two real and single-valued functions with domains X_f, X_g and ranges Y_f and Y_g respectively. Then, the following operations are defined.

(i) Scalar multiplication of a function : $(cf)(x) = cf(x)$, where c is a scalar.

(ii) Addition/subtraction of functions : $(f \pm g)(x) = f(x) \pm g(x)$.

(iii) Multiplication of functions : $(fg)(x) = (gf)(x) = f(x)g(x)$.

(iv) Division of functions : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0.$

EQUAL FUNCTIONS

Two functions f and g are said to be equal functions, if and only if

(i) Domain of f = Domain of g

(ii) Co-domain of f = Co-domain of g

(iii) $f(x) = g(x)\ \forall\ x \in$ their common domain

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TYPES OF FUNCTIONS

(i) One-one function (injectivity)

A function $f : A \to B$ is said to be a one-one function or an injection, if different elements of A have different images in B.

Method to check the injectivity of a function

Step I : Take two arbitrary elements x, y (say) in the domain of f.

Step II : Put $f(x) = f(y)$

Step III : If $f(x) = f(y)$ gives $x = y$ only, then $f : A \to B$ is a one-one function (or an injection), otherwise it is not.

Note : 1. If function is given in the form of ordered pairs and if two ordered pairs do not have same second element then function is one-one.

2. If the graph of the function $y = f(x)$ is given and each line parallel to x-axis cuts the given curve at maximum one point, then function is one-one.

3. Any function, which is entirely increasing or decreasing in the whole of a domain is one-one.

- **Number of one-one functions (injections) :** If A and B are finite sets having m and n elements respectively, then number of one-one functions from A to $B = \begin{cases} {}^nP_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$

(ii) Many-one function

A function $f : A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.

Thus, $f : A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$

Note : 1. If function is given in the form of set of ordered pairs and the second element of atleast two ordered pairs are same, then function is many-one.

2. If the graph of $y = f(x)$ is given and the line parallel to x-axis cuts the curve at more than one point, then function is many-one.

3. If the domain of the function is in one quadrant, then the trigonometric functions are always one-one.

4. If trigonometric function changes its sign in two consecutive quadrants, then it is one-one but if it does not change the sign, then it is many-one.

5. In three consecutive quadrants, trigonometric functions are always many-one.

6. Any continuous function $f(x)$, which has atleast one local maximum or local minimum is many-one.

(iii) Onto function (Surjectivity)

A function $f : A \to B$ is onto if each element of B has its pre-image in A.

In other words,

range of f = co-domain of f.

- **Number of onto functions (surjections) :** If A and B are two sets having m and n elements respectively such that $1 \leq n \leq m$, then number of onto functions from A to B is $\sum_{r=1}^{n} (-1)^{n-r}\ {}^nC_r r^m$.

(iv) Into function

A function $f : A \to B$ is an into function if there exists an element in B having no pre-image in A. In other words, $f : A \to B$ is an into function if it is not an onto function.

Method to find onto or into function

(a) If range = co-domain, then $f(x)$ is onto and if range is a proper subset of the co-domain, then $f(x)$ is into.

(b) Solve $f(x) = y$ by taking x as a function of y *i.e.*, $g(y)$ (say). Now, if $g(y)$ is defined for each $y \in$ co-domain and $g(y) \in$ domain for $y \in$ co-domain, then $f(x)$ is onto.

Note : 1. An into function can be made onto by redefining the co-domain as the range of the original function.

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2. Any polynomial function $f : R \to R$ is onto if degree of f is odd and into if degree of f is even.

(v) One-one Onto function (Bijection)

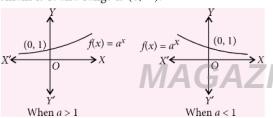
A function $f : A \to B$ is a bijection if it is one-one as well as onto. In other words, a function $f : A \to B$ is a bijection, if for $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$ and for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

- **Number of one-one onto functions (bijections) :** If A and B are finite sets and $f : A \to B$ is a bijection, then A and B have the same number of elements. If A has n elements, then number of bijections from A to B is $n!$

SOME SPECIAL FUNCTIONS

(i) Exponential function

Let $a \neq 1$ be a positive real number. Then $f : R \to (0, \infty)$ defined by $f(x) = a^x$ is called exponential function. Its domain is R and range is $(0, \infty)$.



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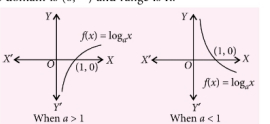
(ii) Logarithmic function

Let $a \neq 1$ be a positive real number. Then $f : (0, \infty) \to R$ defined by $f(x) = \log_a x$ is called logarithmic function. Its domain is $(0, \infty)$ and range is R.

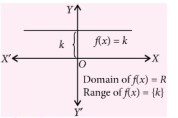


(iii) Explicit and implicit functions

A function is said to be explicit, if it can be expressed directly in terms of the independent variable. If the function can not be expressed directly in terms of the independent variable or variables, then the function is said to be implicit *e.g.* $y = \sin^{-1}x + \log x$ is an explicit function while $x^2 + y^2 = xy$ and $x^3y^2 = (a - x)^2(b - y)^2$ are implicit functions.

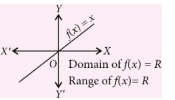
(iv) Constant function

Let k be a fixed real number. Then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.



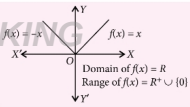
(v) Identity function

The function defined by $f(x) = x$ for all $x \in R$ is called the identity function on R.



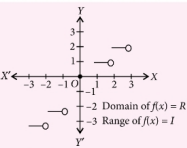
(vi) Modulus function

The function defined by $f(x) = |x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$ is called the modulus function.



(vii) Greatest integer function

The function f defined by $f(x) = [x]$ for all $x \in R$, is called the greatest integer function. Here $[x]$ denotes the greatest integer less than or equal to x for *e.g.* $[1{\cdot}1] = 1$, $[2{\cdot}2] = 2$, $[-0{\cdot}9] = -1$, $[-2{\cdot}1] = -3$ etc.



(viii) Signum function

The function defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

is called the signum function.

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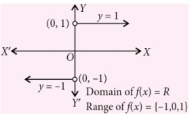
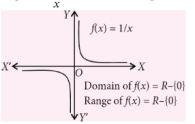


(ix) Reciprocal function

The function that associates each non-zero real number x to the reciprocal $\frac{1}{x}$ is called the reciprocal function.



EVEN AND ODD FUNCTIONS

(i) Even function

If we put $(-x)$ in place of x in the given function and if $f(-x) = f(x)$, $\forall\ x \in$ domain, then $f(x)$ is called even function.

e.g. $f(x) = e^x + e^{-x}$, $f(x) = x^2$, $f(x) = x\sin x$, $f(x) = \cos x$ etc. are all even functions.

(ii) Odd function

If we put $(-x)$ in place of x in the given function and if $f(-x) = -f(x)\ \forall\ x \in$ domain, then $f(x)$ is called odd function.

e.g., $f(x) = e^x - e^{-x}$, $f(x) = \sin x$, $f(x) = x^3$, $f(x) = x\cos x$, $f(x) = x^2 \sin x$ etc. are all odd functions.

Properties of Even and Odd Functions

- The graph of an even function is always symmetric with respect to y-axis.
- The graph of an odd function is always symmetric with respect to origin.
- The sum and difference of two even functions is an even function.
- The product of two even functions is an even function.
- The sum and difference of two odd functions is an odd function.
- The product of two odd functions is an even function.
- The product of an even and an odd function is an odd function
- It is not essential that every function is even or odd. It is possible to have some functions which are neither even nor odd function.
  e.g. $f(x) = x^2 + x^3$ and $f(x) = e^x$.
- The sum of an even and an odd function is neither even nor odd function.

PERIODIC FUNCTION

A function is said to be periodic function, if its each value is repeated after a definite interval. So, a function $f(x)$ will be periodic if there exists a positive real number T such that $f(x + T) = f(x)$, $\forall\ x \in$ domain. Here, the least positive value of T is called the period of the function.

Clearly $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = ...$

e.g. $\sin x$, $\cos x$, $\tan x$ are periodic functions with period 2π, 2π and π respectively.

Some standard results on periodic functions

	Functions	Periods
1.	$\sin^n x$, $\cos^n x$, $\sec^n x$, $\operatorname{cosec}^n x$	$\begin{cases}\pi, \text{ if } n \text{ is even}\\ 2\pi, \text{ if } n \text{ is odd or fraction}\end{cases}$
2.	$\tan^n x$, $\cot^n x$	π, n is even or odd
3.	$\lvert\sin x\rvert$, $\lvert\cos x\rvert$, $\lvert\tan x\rvert$, $\lvert\cot x\rvert$, $\lvert\sec x\rvert$, $\lvert\operatorname{cosec} x\rvert$	π
4.	$x - \lvert x\rvert$	1

COMPOSITE FUNCTION

If $f : A \to B$ and $g : B \to C$ are two functions, then the composite function of f and g, given by $gof : A \to C$ will be defined as $gof(x) = g[f(x)]\ \forall\ x \in A$.

Properties of Composition of Function

- f is even, g is even $\Rightarrow$ fog is even function.
- f is odd, g is odd $\Rightarrow$ fog is odd function.
- f is even, g is odd $\Rightarrow$ fog is even function.
- f is odd, g is even $\Rightarrow$ fog is even function.
- Composition of functions is not commutative, *i.e.*, $fog \neq gof$.
- Composition of functions is associative *i.e.*, $(fog)oh = fo(goh)$.
- $gof(x)$ is simply the g-image of $f(x)$, where $f(x)$ is f-image of elements $x \in A$.
- Function gof will exist only when range of f is the subset of domain of g.
- fog and gof may not be always defined. fog does not exist if range of g is not a subset of domain of f.
- If both f and g are one-one, then fog and gof are also one-one.
- If both f and g are onto, then gof is onto.

INVERSE OF A FUNCTION

If $f : A \to B$ be a one-one onto (bijection) function, then the mapping $f^{-1} : B \to A$ which associates each

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element $b \in B$ with element $a \in A$ such that $f(a) = b$ is called the inverse of the function $f : A \to B$ and denoted by $f^{-1} : B \to A$ as $f^{-1}(b) = a \Rightarrow f(a) = b$

Important Points for Inverse Function

- For the existence of inverse function, it should be one-one and onto.
- Inverse of a bijection is also a bijection function.
- Inverse of a bijection is unique.
- $(f^{-1})^{-1} = f$
- If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = (f^{-1}og^{-1})$
- $fog \neq gof$ but if $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$. Also, $(fog)(x) = (gof)(x) = (x)$.
- If $f : A \to B$ is a bijection then $f^{-1} : B \to A$ is an inverse function of f. $f^{-1}of = I_A$ and $fof^{-1} = I_B$. Here I_A, is an identity function on set A and I_B is an identity function on set B.

PROBLEMS

Single Correct Answer Type

1. If $f(x) = \frac{1}{\sqrt{x+2\sqrt{2x-4}}} + \frac{1}{\sqrt{x-2\sqrt{2x-4}}}$ for $x > 2$, then $f(11) =$

(a) 7/6 (b) 5/6 (c) 6/7 (d) 5/7

2. If for two functions g and f, gof is both injective and surjective, then which of the following is true?

(a) g and f should be injective and surjective
(b) g should be injective and surjective
(c) f should be injective and surjective
(d) none of these

3. The function $f : R \to R$, defined by $f(x) = x^2 + x$ is

(a) one-one onto (b) one-one into
(c) many-one onto (d) many-one into

4. If $f(x, y) = \frac{x}{y}$ where $(x, y) \in R$ and $x, y \neq 0$, then this function is a/an

(a) surjective (b) bijective
(c) one-one (d) none of these

5. If $f(x)$ is periodic function with period T then the function $f(ax + b)$ where $a > 0$, is periodic with period

(a) T/b (b) aT (c) bT (d) T/a

6. Domain of $f(x) = \log|\log x|$ is

(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(0, 1) \cup (1, \infty)$ (d) $(-\infty, 1)$

7. The domain of the function $f(x) = \sin^{-1}[\log_2(x/2)]$ is

(a) $[1, 4]$ (b) $[-4, 1]$
(c) $[-1, 4]$ (d) none of these

8. The domain of the function $f(x) = \frac{\log_2(x+3)}{x^2+3x+2}$ is

(a) $R - \{-1, -2\}$ (b) $(-2, \infty)$
(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

9. The domain of the function $f(x) = \sqrt{\log \frac{1}{|\sin x|}}$ is

(a) $R - \{2n\pi,\ n \in I\}$ (b) $R - \{n\pi,\ n \in I\}$
(c) $R - \{-\pi, \pi\}$ (d) $(-\infty, \infty)$

10. The domain of the function $f(x) = \log(\sqrt{x-4} + \sqrt{6-x})$ is

(a) $[4, \infty)$ (b) $(-\infty, 6]$
(c) $[4, 6]$ (d) none of these

11. The domain of the function $f(x) = \log_{3+x}(x^2 - 1)$ is

(a) $(-3, -1) \cup (1, \infty)$ (b) $[-3, -1] \cup [1, \infty)$
(c) $(-3, -2) \cup (-2, -1) \cup (1, \infty)$
(d) $[-3, -2) \cup (-2, -1) \cup [1, \infty)$

12. If n is an integer, then the domain of the function $f(x) = \sqrt{\sin 2x}$ is

(a) $\left[n\pi - \frac{\pi}{2}, n\pi\right]$ (b) $\left[n\pi, n\pi + \frac{\pi}{2}\right]$
(c) $[(2n-1)\pi, 2n\pi]$ (d) $[2n\pi, (2n+1)\pi]$

13. Domain of the function $\sqrt{\log[(5x - x^2)/6]}$ is

(a) $(2, 3)$ (b) $[2, 3]$ (c) $[1, 2]$ (d) $[1, 3]$

14. The domain of the function $y = \frac{1}{\sqrt{|x| - x}}$ is

(a) $(-\infty, 0)$ (b) $(-\infty, 0]$ (c) $(-\infty, -1)$ (d) $(-\infty, \infty)$

15. The function $f : R \to R$, $f(x) = x^2\ \forall x \in R$ is

(a) injective but not surjective
(b) surjective but not injective
(c) injective as well as surjective
(d) neither injective nor surjective

16. If $f(x) = \frac{x}{1+x}$, then $f^{-1}(x)$ is equal to

(a) $\frac{(1+x)}{x}$ (b) $\frac{1}{(1+x)}$ (c) $\frac{(1+x)}{(1-x)}$ (d) $\frac{x}{(1-x)}$

17. The inverse of the function $\frac{10^x - 10^{-x}}{10^x + 10^{-x}}$ is

(a) $\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$ (b) $\frac{1}{2}\log_{10}\left(\frac{1-x}{1+x}\right)$

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(c) $\frac{1}{4}\log_{10}\left(\frac{2x}{2-x}\right)$ (d) none of these

18. If $f(x)=\frac{x}{\sqrt{1+x^2}}$, then $(fofof)(x)$ =

(a) $\frac{3x}{\sqrt{1+x^2}}$ (b) $\frac{x}{\sqrt{1+3x^2}}$

(c) $\frac{x}{\sqrt{1+x^2}}$ (d) none of these

19. If $g(x) = x^2 + x - 2$ and $\frac{1}{2}(gof)(x) = 2x^2 - 5x + 2$, then $f(x)$ is

(a) $2x - 3$ (b) $2x + 3$

(c) $2x^2 + 3x + 1$ (d) $2x^2 - 3x - 1$

20. If $f(x) = (x + 1)^2$, $g(x) = x^2 + 1$ defined over real numbers, then $(fog)(-3)$ equals

(a) 121 (b) 112

(c) 211 (d) none of these

21. Let $f(x) = ax + b$ and $g(x) = cx + d$, $a \neq 0$, $c \neq 0$. Assume $a = 1$, $b = 2$. If $(fog)(x) = (gof)(x)$ for all x, then what can you say about c and d?

(a) c and d both arbitrary (b) $c = 1$, d arbitrary

(c) c arbitrary, $d = 1$ (d) none of these

22. Let $g(x) = 1 + x - [x]$ and $f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$, then for all x, $f(g(x))$ is equal to

(a) x (b) 1 (c) $f(x)$ (d) $g(x)$

23. If $f(x)=\frac{\alpha x}{x+1}, x \neq -1$. Then, value of α for which $f(f(x)) = x$ is

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

24. Let $f : (-1, 1) \to B$ be a function defined by $f(x) = \tan^{-1}\frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$

25. If $f(x) = 2x^6 + 3x^4 + 4x^2$, then $f'(x)$ is

(a) an even function

(b) an odd function

(c) neither even nor odd

(d) can't be determined

26. If a real valued function $f(x)$ satisfies the functional equation $f(x - y) = f(x) f(y) - f(a - x) f(a + y)$, where a is a given constant and $f(0) = 1$ then, $f(2a - x)$ is equal to

(a) $f(a) + f(a - x)$ (b) $f(-x)$

(c) $-f(x)$ (d) $f(x)$

27. If $g : [-2, 2] \to R$ where $g(x) = x^3 + \tan x + \left[\frac{x^2+1}{P}\right]$ is an odd function where [·] denotes greatest integer function, then the value of parameter P is

(a) $-5 < P < 5$ (b) $P < 5$

(c) $P > 5$ (d) none of these

28. If $f(x)=\begin{cases}x, & \text{when } x \text{ is rational}\\ 0, & \text{when } x \text{ is irrational}\end{cases}$

and $g(x)=\begin{cases}0, & \text{when } x \text{ is rational}\\ x, & \text{when } x \text{ is irrational}\end{cases}$, then $(f - g)$ is

(a) one-one onto (b) one-one but not onto

(c) not one-one but onto

(d) neither one-one nor onto

29. Range of the function $f : R \to R$ given by $f(x)=\frac{x^2}{x^2+1}$ is

(a) $(-1, 0)$ (b) $(-1, 1)$ (c) $[0, 1)$ (d) $(0, 1)$

30. The function f satisfies the functional equation $3f(x)+2f\left(\frac{x+59}{x-1}\right)=10x+30$ for all real $x \neq 1$. The value of $f(7)$ is

(a) 8 (b) 4 (c) -8 (d) 11

Multiple Correct Answer Type

31. If $f(x_1) - f(x_2) = f\left(\frac{x_1 - x_2}{1 - x_1 x_2}\right)$ for $x_1, x_2 \in [-1, 1]$, then $f(x)$ is

(a) $\log\frac{(1-x)}{(1+x)}$ (b) $\tan^{-1}\frac{(1-x)}{(1+x)}$

(c) $\log\frac{(1+x)}{(1-x)}$ (d) $\tan^{-1}\frac{(1+x)}{(1-x)}$

32. Let $f(x)=\frac{\sin x}{\sqrt{1+\tan^2 x}}+\frac{\cos x}{\sqrt{1+\cot^2 x}}$, then

(a) fundamental period of $f(x)$ is 2π

(b) fundamental period of $f(x)$ is π

(c) domain of $f(x)$ is R

(d) range of $f(x)$ is $[-1, 1]$

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33. Let $f : R \to R$ be a function defined by $f(x+1)=\frac{f(x)-5}{f(x)-3}, \forall x \in R$. Then which of the following statement(s) is/are true?

(a) $f(x+2)=f(x)$ (b) $f(x+4)=f(x)$

(c) $f(x+6)=f(x)$ (d) $f(x+8)=f(x)$

34. Let f be a function defined by $f(x)=\frac{x-5}{x-3}$, $x \neq 3, 2$; $f^k(x)$ denote the composition of f with itself taken k times *i.e.*, $f^3(x)=f(f(f(x)))$. Then,

(a) $f^{2012}(2009)=2009$ (b) $f^{2009}(2010)=\frac{2006}{2007}$

(c) $f^{2009}(2011)=\frac{1003}{1004}$ (d) $f^{2012}(2012)=2012$

Comprehension Type

Paragraph for Q. No. 35-37

A function $f(x)$ is said to be even iff $f(-x)=f(x)$ and odd iff $f(-x)=-f(x)$. Also $f(x)$ is said to be periodic iff $f(x+T)=f(x)$, $T>0$ and the least positive value of T is called its period.

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35. If $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$, then $f(x)$ is

(a) an even function (b) an odd function

(c) neither even nor odd (d) both even and odd

36. $f(x)$ is an even polynomial function. Then $\sin(f(x)-3x)$ is

(a) an even function (b) an odd function

(c) neither even nor odd

(d) periodic with period $\pi/2$

37. If $f(x)$ and $g(x)$ be two functions with all real numbers as their domains, then

$h(x)=[f(x)+f(-x)][g(x)-g(-x)]$ is

(a) an even function when f is even and g is odd

(b) an even function when both f and g are odd

(c) always an odd function

(d) always an even function

Paragraph for Q. No. 38-39

Let us consider the two non empty sets X and Y such that $X=\{1, 2, 3, 4, 5, 6, 7\}$ and $Y=\{1, 2, 3, 4, 5, 6\}$.

38. The number of many one functions from X to Y, is

(a) 6^7 (b) 7^6 (c) 7^6-6^7 (d) $6^7-6!$

39. The number of functions from X to Y such that $f(x_i) \neq x_j \ \forall \ x_i \in \{1, 2, 3, \ldots, 7\}$, is

(a) $6(5)^6$ (b) $5(6)^5$

(c) $5(7)^6$ (d) none of these

Matrix-Match Type

40. Match the following:

	Column- I		Column- II
A.	If f is a function such that $f(0)=2$, $f(1)=3$ and $f(x+2)=2f(x)-f(x+1)$, then $f(5)$ is equal to	(p)	4
B.	If $f(x)=\begin{cases}x^2, & \text{for } x \geq 0\\ x, & \text{for } x<0\end{cases}$, then $f(\sqrt{13})=$	(q)	3
C.	If $f(x)+2f(1-x)=x^2+2$ for all $x \in R$, then $f(5)$ is	(r)	12
D.	If $f(x)=\frac{4^x}{4^x+2}$ for all $x \in R$, then $\sum_{k=1}^{6} f\left(\frac{k}{7}\right)$ equals	(s)	2
		(t)	13

41. Match the following:

	Column-I		Column-II
A.	Let $f : R \to R$ satisfies $f(x+y)+f(x-y)=2f(x)f(y) \ \forall \ x, y \in R$ and $f(0) \neq 0$, then $f(x)$ is	(p)	even
B.	Let $f : R \to R$ is defined by $f(x)=\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}$, then $f(x)$ is	(q)	odd
C.	Let $f : R \to R$ be a polynomial function satisfying $f(x)f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$ and $f(3)=28$, then $f(x)$ is	(r)	into
D.	Let $f : R \to R$ is defined by $f(x)=2x+\sin x$, then $f(x)$ is	(s)	many-one
		(t)	bijective

Integer Answer Type

42. If n is a natural number and $1 \leq n \leq 100$, then the number of solutions of $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{5}\right]=\frac{n}{2}+\frac{n}{3}+\frac{n}{5}$ (where [.] denotes the greatest integer function) is

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43. If f is a function such that $f(x) + 2f(1 - x) = x^2 + 1$. then value of $f(3)$ is

44. The number of solutions of the equation $x^2 + [x] - 4x + 3 = 0$ is where ([.] denotes the greatest integer function)

45. If the function $f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$

and $g(x) = \begin{cases} x^2, & -1 \le x \le 2 \\ x+2, & 2 \le x \le 3 \end{cases}$ then the number of

roots of the equation $f(g(x)) = 2$ is

SOLUTIONS

1. (c) : $f(x) = \dfrac{1}{\sqrt{x+2\sqrt{2x-4}}} + \dfrac{1}{\sqrt{x-2\sqrt{2x-4}}}$

$\therefore \quad f(11) = \dfrac{1}{\sqrt{11+2\sqrt{18}}} + \dfrac{1}{\sqrt{11-2\sqrt{18}}}$

$= \dfrac{1}{3+\sqrt{2}} + \dfrac{1}{3-\sqrt{2}} = \dfrac{3-\sqrt{2}}{7} + \dfrac{3+\sqrt{2}}{7} = \dfrac{6}{7}$

2. (a) : As *gof* is bijective, then g and f are also bijective.

3. (d) : $\because f(0) = f(-1) = 0 \Rightarrow f(x)$ is many one.
Also, there is no pre-image of -1. Hence, $f(x)$ is into function.

4. (a) : $\because \quad f(\alpha x, \alpha y) = \dfrac{\alpha x}{\alpha y} = \dfrac{x}{y} (\alpha \ne 0)$.

Hence, $f(x, y)$ is not an injective function.

5. (d) : $f\left(a\left(x + \dfrac{T}{a}\right) + b\right) = f(ax + b + T) = f(ax + b)$.

6. (c) : $f(x) = \log|\log x|$, $f(x)$ is defined if $|\log x| > 0$ and $x > 0$.
i.e., $x > 0$ and $x \ne 1$ $\quad (\because \ |\log x| > 0, \text{ if } x \ne 1)$
$\therefore \quad x \in (0, 1) \cup (1, \infty)$

7. (a) : We have, $f(x) = \sin^{-1}[\log_2(x/2)]$.
Domain of $\sin^{-1}x$ is $[-1, 1]$

$\Rightarrow \quad -1 \le \log_2(x/2) \le 1 \Rightarrow \dfrac{1}{2} \le \dfrac{x}{2} \le 2 \Rightarrow 1 \le x \le 4$
$\therefore \quad x \in [1, 4]$.

8. (d) : Here, $x + 3 > 0$ and $x^2 + 3x + 2 \ne 0$
$\therefore \quad x > -3$ and $(x + 1)(x + 2) \ne 0$, *i.e.*, $x \ne -1, -2$
$\therefore \quad$ Domain $= (-3, \infty) - \{-1, -2\}$.

9. (b) : We have, $f(x) = \sqrt{\log \dfrac{1}{|\sin x|}}$

$\Rightarrow \quad \sin x \ne 0 \Rightarrow x \ne n\pi, n \in I$
$\therefore \quad$ Domain of $f(x) = R - \{n\pi, n \in I\}$.

10. (c) : $f(x) = \log(\sqrt{x-4} + \sqrt{6-x})$

$\Rightarrow \quad \sqrt{x-4} + \sqrt{6-x} > 0$
$\therefore \quad$ Domain of $f(x) = [4, 6]$.

11. (c) : We have, $f(x) = \log_{3+x}(x^2 - 1)$
$\therefore \quad x^2 > 1 \Rightarrow x < -1$ or $x > 1$ and $3 + x > 0$
$\Rightarrow \quad x > -3$. Also, $x \ne -2$
$\therefore \quad D_f = (-3, -2) \cup (-2, -1) \cup (1, \infty)$.

12. (b) : $\because \sin 2x$ can't be negative
$\therefore \quad$ Domain of function $\sqrt{\sin 2x}$ is $\left[n\pi, n\pi + \dfrac{\pi}{2}\right]$.

13. (b) : $\log\left\{\dfrac{5x - x^2}{6}\right\} \ge 0 \Rightarrow \dfrac{5x - x^2}{6} \ge 1$

or $\quad x^2 - 5x + 6 \le 0 \Rightarrow 2 \le x \le 3$

14. (a) : We have, $y = \dfrac{1}{\sqrt{|x| - x}} \Rightarrow |x| - x > 0$ or $|x| > x$

But $|x| = x$ for all positive x and $|x| > x$ for all negative x.
So, domain will be $(-\infty, 0)$.

15. (d) : The mapping is many-one into.

16. (d) : Let $y = f(x) = \dfrac{x}{1+x} \Rightarrow x = f^{-1}(y)$
$\therefore \quad y = \dfrac{x}{1+x} \Rightarrow y + yx = x \Rightarrow x = \dfrac{y}{1-y}$

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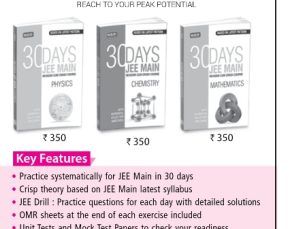

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$\Rightarrow \quad f^{-1}(y)=\frac{y}{1-y} \Rightarrow f^{-1}(x)=\frac{x}{1-x}$.

17. (a) : Let $y=f(x)=\frac{10^x-10^{-x}}{10^x+10^{-x}}$

$\Rightarrow \quad x=\frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right)$

$\Rightarrow \quad f^{-1}(y)=\frac{1}{2}\log_{10}\left(\frac{1+y}{1-y}\right)$ $(\because y=f(x) \Rightarrow x=f^{-1}(y))$

$\therefore \quad f^{-1}(x)=\frac{1}{2}\log_{10}\left(\frac{1+x}{1-x}\right)$.

18. (b) : $(fofof)(x)=(fof)(f(x))=(fof)\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$=f\left[\frac{\left(\frac{x}{\sqrt{1+x^2}}\right)}{\sqrt{1+\frac{x^2}{1+x^2}}}\right]=f\left(\frac{x\sqrt{1+x^2}}{\sqrt{1+x^2}\sqrt{1+2x^2}}\right)$$

$$=f\left(\frac{x}{\sqrt{1+2x^2}}\right)=\frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}}=\frac{x}{\sqrt{1+3x^2}}.$$

19. (a) : Given, $\frac{1}{2}(gof)(x)=2x^2-5x+2$

or $\quad \frac{1}{2}g[f(x)]=2x^2-5x+2$

$\therefore \quad [\{f(x)\}^2+\{f(x)\}-2]=2[2x^2-5x+2]$

$\Rightarrow \quad [f(x)]^2+f(x)-(4x^2-10x+6)=0$

$\therefore \quad f(x)=\frac{-1\pm\sqrt{1+4(4x^2-10x+6)}}{2}$

$=\frac{-1\pm\sqrt{(16x^2-40x+25)}}{2}=\frac{-1\pm(4x-5)}{2}=2x-3, 2-2x$

20. (a) : $f(x)=(1+x)^2$ and $g(x)=x^2+1$

Also, $fog(-3)=f[g(-3)]=f[9+1]=f[10]=[11]^2=121$.

21. (b) : $(fog)(x)=f(g(x))=a(cx+d)+b$

and $(gof)(x)=g(f(x))=c(ax+b)+d$

Given that, $(fog)(x)=(gof)(x)$

At $a=1$, $b=2$, we have

$cx+d+2=cx+2c+d$

$\Rightarrow \quad c=1$ and d is arbitrary.

22. (b) : Here $g(x)=1+n-n=1, x=n \in Z$

$g(x)=1+n+k-n=1+k, x=n+k$

(where $n \in Z$, $0<k<1$)

Clearly, $g(x)>0$ for all x. So, $f(g(x))=1$ for all x.

23. (d) : $f(f(x))=\frac{\alpha f(x)}{f(x)+1}=\frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\left(\frac{\alpha x}{x+1}+1\right)}=\frac{\alpha^2\cdot x}{\alpha x+x+1}$

Now, $x=\frac{\alpha^2\cdot x}{(\alpha+1)x+1} \Rightarrow x[(\alpha+1)x+1-\alpha^2)]=0$

or $\quad (\alpha+1)x^2+(1-\alpha^2)x=0$.

This should hold for all x.

$\Rightarrow \quad \alpha+1=0, 1-\alpha^2=0 \Rightarrow \alpha=-1$.

24. (b) : For $-1<x<1$, $\tan^{-1}\frac{2x}{1-x^2}=2\tan^{-1}x$

Range of $f(x)=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore \quad$ Co-domain of function $=B=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

25. (b) : We have, $f(x)=2x^6+3x^4+4x^2$

$f(-x)=2(-x)^6+3(-x)^4+4(-x)^2=f(x)$

$\therefore \quad f(x)$ is an even function and derivative of an even function is always odd.

26. (c) : $f(a-(x-a))=f(a)f(x-a)-f(0)f(x)$ ...(i)

Put $x=0$, $f(0)=(f(0))^2-[f(a)]^2 \Rightarrow f(a)=0$

From (i), $f(2a-x)=-f(x)$. $[\because f(0)=1]$

27. (c) : Given, $g(x)=x^3+\tan x+\left[\frac{x^2+1}{P}\right]$

$\therefore \quad g(-x)=-x^3-\tan x+\left[\frac{x^2+1}{P}\right]$

Now, $g(x)+g(-x)=0$ $[\because g(x)$ is an odd function$]$

$\therefore \quad x^3+\tan x+\left[\frac{x^2+1}{P}\right]-x^3-\tan x+\left[\frac{x^2+1}{P}\right]=0$

$\Rightarrow \quad 2\left[\frac{x^2+1}{P}\right]=0 \Rightarrow 0\le\frac{x^2+1}{P}<1$

Since, $x \in [-2, 2] \Rightarrow \quad 0\le\frac{5}{P}<1 \Rightarrow P>5$.

28. (a) : $(f-g)(x)=\begin{cases}x, & x\in Q\\ -x, & x\notin Q\end{cases}$

29. (c) : Let $y=\frac{x^2}{x^2+1}$

$\Rightarrow \quad (y-1)x^2+0x+y=0, y\neq 1$

For real values of x, $D\ge 0$

$\Rightarrow \quad -4y(y-1)\ge 0 \Rightarrow y(y-1)\le 0$

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$\Rightarrow \quad y \in [0, 1) \quad \left[\because \quad 0 \le \frac{x^2}{x^2+1} < 1\right]$

30. (b) : We have, $3f(x) + 2f\left(\frac{x+59}{x-1}\right) = 10x + 30$

For $x = 7$, $3f(7) + 2f(11) = 70 + 30 = 100$

For $x = 11$, $3f(11) + 2f(7) = 140$

$\therefore \quad \frac{f(7)}{-20} = \frac{f(11)}{-220} = \frac{-1}{9-4} \Rightarrow f(7) = 4.$

31. (a, b, c) : When $x_1 = -1$ and $x_2 = 1$, then

$f(-1) - f(1) = f\left[\frac{-1-1}{1+1(1)}\right] = f(-1) \Rightarrow f(1) = 0,$

which is satisfied when $f(x) = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

When $x_1 = x_2 = 0$, then

$f(0) - f(0) = f\left[\frac{0-0}{1-0}\right] = f(0) \Rightarrow f(0) = 0 \qquad ...(i)$

Also, when $x_1 = -1$ and $x_2 = 0$ then

$f(-1) - f(0) = f\left(\frac{-1-0}{1-0}\right) = f(-1) \Rightarrow f(0) = 0 \qquad ...(ii)$

Both (i) and (ii) are satisfied when

$f(x) = \log\left(\frac{1-x}{1+x}\right)$ and $f(x) = \log\left(\frac{1+x}{1-x}\right)$

32. (a, d) : $f(x) = \begin{cases} \sin 2x, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \\ -\sin 2x, & \pi < x < \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} < x < 2\pi \end{cases}$

So, $f(x)$ is periodic with fundamental period 2π and range $[-1, 1]$.

33. (b, d) : Given, $f(x+1) = \frac{f(x)-5}{f(x)-3} \qquad ...(i)$

$\Rightarrow \quad f(x) = \frac{3f(x+1)-5}{f(x+1)-1}$

Replace 'x' by $x - 1$, $f(x-1) = \frac{3f(x)-5}{f(x)-1} \qquad ...(ii)$

Replace 'x' by $x + 1$ in (i)

$f(x+2) = \frac{f(x+1)-5}{f(x+1)-3} = \frac{2f(x)-5}{f(x)-2} \qquad ...(iii)$

Similarly, $f(x-2) = \frac{2f(x)-5}{f(x)-2}$

$\therefore \quad f(x+2) = f(x-2) \Rightarrow f(x+4) = f(x)$

and $f(x+8) = f(x+4) = f(x)$

34. (a, c, d) : $f(x) = \frac{x-5}{x-3} \Rightarrow f^2(x) = \frac{2x-5}{x-2} = g(x)$

Then, $g^{2n}(x) = x$ and $g^{2n+1} = g(x)$

$f^{2012}(2009) = g^{1006}(2009) = 2009$

$f^{2009}(2010) = f[g^{1004}(2010)] = f(2010) = \frac{2005}{2007}$

$f^{2009}(2011) = f[g^{1004}(2011)] = f(2011) = \frac{2006}{2008}$

$f^{2012}(2012) = g^{1006}(2012) = 2012$

35. (c) : $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

$f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1 = \frac{xe^x}{e^x - 1} - \frac{x}{2} + 1 = \frac{x}{1 - e^{-x}} - \frac{x}{2} + 1$

Thus, $f(x)$ is neither even nor odd.

36. (c) : Let $g(x) = \sin[f(x) - 3x]$

$g(-x) = \sin[f(-x) + 3x] = \sin[f(x) + 3x]$

37. (c) : $h(-x) = [f(-x) + f(x)][g(-x) - g(x)] = -h(x)$

38. (a) : Number of many-one functions $= 6^7$.

39. (a) : 7 can be associated to $y \in Y$ in 6 ways and 1, 2, 3, 4, 5, 6 can be associated in 5 ways each.

$\therefore \quad$ The total number of functions required $= 6(5)^6$.

40. $\mathbf{A \to (t); B \to (t); C \to (q); D \to (q)}$

(A) Given $f(0) = 2$, $f(1) = 3$

and $f(x + 2) = 2f(x) - f(x + 1)$

Put $x = 0$; $f(2) = 2f(0) - f(1) = 2 \times 2 - 3 = 1$

Put $x = 1$; $f(3) = 2f(1) - f(2) = 2 \times 3 - 1 = 5$

Put $x = 2$; $f(4) = 2f(2) - f(3) = 2 \times 1 - 5 = -3$

Put $x = 3$; $f(5) = 2f(3) - f(4) = 2 \times 5 - (-3) = 13$

(B) $\sqrt{13} > 0 \Rightarrow f\left(\sqrt{13}\right) = \left(\sqrt{13}\right)^2 = 13$

(C) Given $f(x) + 2f(1 - x) = x^2 + 2, \forall x \in R \qquad ...(i)$

Replace x by $1 - x$, we get

$f(1-x) + 2f(x) = (1-x)^2 + 2 \qquad ...(ii)$

Solving (i) & (ii), we get $f(x) = \frac{(x-2)^2}{3}$

$\therefore \quad f(5) = 3.$

(D) Given, $f(x) = \frac{4^x}{4^x + 2}$

$\Rightarrow \quad f(1-x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{4^x + 2}$

$\therefore \quad f(x) + f(1 - x) = 1$

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$\Rightarrow\ f\left(\frac{1}{7}\right)+f\left(\frac{6}{7}\right)=1\ ;\ f\left(\frac{2}{7}\right)+f\left(\frac{5}{7}\right)=1$

and $f\left(\frac{3}{7}\right)+f\left(\frac{4}{7}\right)=1\ \Rightarrow\ \sum_{k=1}^{6} f\left(\frac{k}{7}\right)=1+1+1=3$

41. $\mathbf{A \to (p, r, s)\ ;\ B \to (r,s)\ ;\ C \to (t)\ ;\ D \to (q, t)}$

(A) Put $x = y = 0$, then $2f(0) = 2\{f(0)\}^2 \Rightarrow f(0) = 1$

Now put $x = 0$, then

$f(y) + f(-y) = 2f(0)\, f(y)$

$\therefore\ f(y) = f(-y) \Rightarrow f(x)$ is even function

$\Rightarrow\ f(x)$ is many one and cannot be onto.

(B) If $x \le 0$ then $f(x) = 0 \Rightarrow f(x)$ is many one and neither even nor odd. Clearly, $f(x)$ is not onto.

(C) $f(x)$ is either $-x^n + 1$ or $x^n + 1$. But $f(3) = 28$

$\therefore\ f(x) = x^3 + 1$ which is neither even nor odd but one-one and onto both.

(D) Obviously, $f(x)$ is odd and $f'(x) = 2 + \cos x > 0$, so $f(x)$ is one-one.

Also, $f(x) \to \infty$ if $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$ so $f(x)$ is onto.

42. (3) : From the given equation, we have

$$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$$

$$\Rightarrow \left\{\frac{n}{2}\right\}+\left\{\frac{n}{3}\right\}+\left\{\frac{n}{5}\right\}=0;\ \text{where } \{.\} \text{ is a fractional part .}$$

But each of the fractional part function is positive and their sum is zero. Hence, each of the fractional part function is zero. Consequently, each of $\frac{n}{2}, \frac{n}{3}, \frac{n}{5}$ is an integer. The L.C.M. of 2, 3, 5 is 30. Therefore we can take $n = 30k$, where k is an integer.

Hence, the number of solutions such that $1 \le n \le 100$ is 3 (n = 30, 60 and 90).

43. (0) : $f(x) + 2f(1 - x) = x^2 + 1$ …(i)

$\therefore\ f(1 - x) + 2f(x) = (1 - x)^2 + 1 = x^2 - 2x + 2$ …(ii)

Solving (i) and (ii), we get $3f(x) = x^2 - 4x + 3$

$\therefore\ f(3) = 0$

44. (0) : From the given equation, we get

$x^2 + (x - f) - 4x + 3 = 0$, where f = fractional part function such that $0 \le f < 1$

$\Rightarrow\ (x^2 - 3x + 3) - f = 0 \Rightarrow f = x^2 - 3x + 3$

But $0 \le f < 1$. Therefore $0 \le x^2 - 3x + 3 < 1$ …(i)

Now, $x^2 - 3x + 3 = 0$ gives $x = \frac{3 \pm \sqrt{9-12}}{2}$ (imaginary)

Again from $x^2 - 3x + 3 < 1$, we get $x^2 - 3x + 2 < 0$

$\Rightarrow\ (x - 2)(x - 1) < 0 \Rightarrow 1 < x < 2$.

Thus the inequality (i) is satisfied if $1 < x < 2$

$\Rightarrow\ [x] = 1$.

Putting $[x] = 1$ in the given equation, we get

$x^2 + 1 - 4x + 3 = 0 \Rightarrow (x - 2)^2 = 0 \Rightarrow x = 2$

But $x = 2$ does not satisfy the inequality $1 < x < 2$.

$\therefore$ The given equation has no solution.

45. (4) : $f(g(x)) = \begin{cases} x^2+1, & -1 \le x \le 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

Now, $f(g(x)) = 2 \Rightarrow x^2 + 1 = 2$ and $2x^2 + 1 = 2$

$\Rightarrow\ x = \pm 1$ and $x = \pm\frac{1}{\sqrt{2}}$

$\therefore$ Number of solutions = 4.

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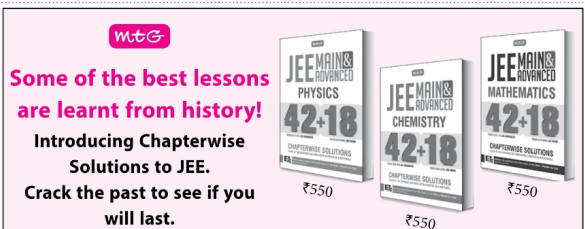

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Limits Book Corner Official

CONCEPT MAP

LIMITS

Some Standard Limits

- $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, n being a positive integer
- $\lim_{x\to 0}\frac{\sin x}{x}=1$ • $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$ • $\lim_{x\to 0}\cos x=1$
- $\lim_{x\to 0}\frac{\tan x}{x}=1$ • $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$ • $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

Theorems

- Let f and g be two real valued functions with the same domain such that $f(x)\le g(x)$ for all x in the domain of f and g. For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)\le \lim_{x\to a} g(x)$.
- **Sandwich Theorem :** Let f, g and h be real functions such that $f(x)\le g(x)\le h(x)$ for all $x\in$ {dom $f(x)\cap$ dom $g(x)\cap$ dom $h(x)$}. For some real number a, if $\lim_{x\to a} f(x)=l=\lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x)=l$.

Algebra of Limits

Let $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$. Then

- $\lim_{x\to a}(f(x)\pm g(x))=l\pm m$
- $\lim_{x\to a} kf(x)=kl$, k is constant
- $\lim_{x\to a} f(x)g(x)=lm$
- $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{l}{m}$, $m\ne 0$
- $\lim_{x\to a}|f(x)|=|l|$
- $\lim_{x\to a} f(x)^{g(x)}=l^m$, $l\ne 0$ and $m\ne 0$
- $\lim_{x\to a} e^{f(x)}=e^l$
- $\lim_{x\to a}(\ln f(x))=\ln l$, $l>0$

L' Hospital's Rule

Let $f(a)=0$, $g(a)=0$ and $f(x)$, $g(x)$ are differentiable functions with derivatives $f'(x)$, $g'(x)$, then $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$

$$=\lim_{x\to a}\frac{f(x)-f(a)}{\frac{g(x)-g(a)}{x-a}}=\lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}=\frac{f'(a)}{g'(a)},\ g'(a)\ne 0$$

or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f''(a)}{g''(a)}$, if $f'(a)=g'(a)=0$ and so on.

Definition

- Limit of a function at $x=a$ is the expected value of $f(x)$ when x tends to a. It is denoted by $\lim_{x\to a} f(x)$.
- Limit of a function $y=f(x)$ at a point 'a' exists iff L.H.L = R.H.L. where L.H.L = $\lim_{x\to a^-} f(x)$. (Left hand limit) means the expected value of $f(x)$ as dictated by the points to the left of a and R.H.L = $\lim_{x\to a^+} f(x)$ (Right hand limit) means the expected value of $f(x)$ as dictated by the points to the right of a.

Evaluation of Limits by Series

Sometimes, $\lim_{x\to a} f(x)$ can be evaluated by using the following series:

- $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$
- $a^x=1+(\ln a)\frac{x}{1!}+(\ln a)^2\frac{x^2}{2!}+(\ln a)^3\frac{x^3}{3!}+\dots$
- $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots,-1<x<1$
- $\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$
- $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots$
- $\tan x=x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots$
- $x\cot x=1-\frac{x^2}{3}-\frac{x^4}{45}-\dots$
- $\sec x=1+\frac{x^2}{2}+\frac{5}{24}x^4+\dots$
- $x\operatorname{cosec} x=1+\frac{x^2}{6}+\frac{7x^4}{360}+\dots$
- $\sin^{-1}x=x+\frac{1}{2}\frac{x^3}{3}+\frac{1}{2}\frac{3}{4}\frac{x^5}{5}+\frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7}+\dots$
- $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots$
- $(1+x)^{1/x}=e\left(1-\frac{x}{2}+\frac{11x^2}{24}-\dots\right)$

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INVERSE TRIGONOMETRIC FUNCTIONS

Class XI Class XII

Function	Domain	Range
$\sin^{-1}x$	$[-1,1]$	$[-\pi/2,\pi/2]$
$\cos^{-1}x$	$[-1,1]$	$[0,\pi]$
$\tan^{-1}x$	R	$(-\pi/2,\pi/2)$
$\cot^{-1}x$	R	$(0,\pi)$
$\operatorname{cosec}^{-1}x$	$R-(-1,1)$	$[-\pi/2,\pi/2]-\{0\}$
$\sec^{-1}x$	$R-(-1,1)$	$[0,\pi]-\{\pi/2\}$

- $\sin^{-1}(-x)=-\sin^{-1}x,\ x\in[-1,1]$
- $\cos^{-1}(-x)=\pi-\cos^{-1}x,\ x\in[-1,1]$
- $\tan^{-1}(-x)=-\tan^{-1}x,\ x\in R$
- $\cot^{-1}(-x)=\pi-\cot^{-1}x,\ x\in R$
- $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x,\ |x|\ge 1$
- $\sec^{-1}(-x)=\pi-\sec^{-1}x,\ |x|\ge 1$

- $\sin^{-1}x+\cos^{-1}x=\pi/2$
- $\tan^{-1}x+\cot^{-1}x=\pi/2$
- $\operatorname{cosec}^{-1}x+\sec^{-1}x=\pi/2$

- $\sin^{-1}(1/x)=\operatorname{cosec}^{-1}x;\ x\ge 1$ or $x\le -1$
- $\cos^{-1}(1/x)=\sec^{-1}x;\ x\ge 1$ or $x\le -1$
- $\tan^{-1}(1/x)=\cot^{-1}x;\ x>0$

- $3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3), & \text{if } \frac{1}{2}<x\le 1\\ -\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x<-\frac{1}{2}\end{cases}$
- $3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x), & \text{if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x), & \text{if } -1\le x\le -\frac{1}{2}\end{cases}$
- $3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\frac{1}{\sqrt{3}}\end{cases}$

- $\sin(\sin^{-1}x)=x\ \forall\ x\in[-1,1]$
- $\cos(\cos^{-1}x)=x\ \forall\ x\in[-1,1]$
- $\tan(\tan^{-1}x)=x\ \forall\ x\in R$
- $\cot(\cot^{-1}x)=x\ \forall\ x\in R$
- $\operatorname{cosec}(\operatorname{cosec}^{-1}x)=x\ \forall\ x\in R-(-1,1)$
- $\sec(\sec^{-1}x)=x\ \forall\ x\in R-(-1,1)$

- $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$, if $-1\le x,y\le 1$ and $x^2+y^2\le 1$ or if $xy<0$ and $x^2+y^2>1$
- $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}$, if $-1\le x,y\le 1$ and $x^2+y^2\le 1$ or if $xy>0$ and $x^2+y^2>1$
- $\cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})$, if $-1\le x,y\le 1$ and $x+y\ge 0$
- $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\}$, if $-1\le x,y\le 1$ and $x\le y$
- $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),xy<1$
- $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right),xy>-1$

- $2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}),|x|\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}),\frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}),-1\le x\le-\frac{1}{\sqrt{2}}\end{cases}$
- $2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1),0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1),-1\le x\le 0\end{cases}$
- $2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right),-1<x<1\\ \sin^{-1}\left(\frac{2x}{1+x^2}\right),|x|\le 1\\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right),x\ge 0\end{cases}$

- $\frac{1}{2}\sin^{-1}x=\tan^{-1}\left(\frac{1-\sqrt{1-x^2}}{x}\right)$
- $\frac{1}{2}\cos^{-1}x=\tan^{-1}\left(\frac{1-x}{\sqrt{1-x^2}}\right)$
- $\frac{1}{2}\tan^{-1}x=\tan^{-1}\left(\frac{\sqrt{x^2+1}-1}{x}\right)$

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CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the reduced syllabus and marking scheme issued by CBSE for the academic session 2020-21.

Series 4

Continuity and Differentiability and Application of Derivatives

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

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(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION-A

(Q.1 - Q.10) are multiple choice type questions. Select the correct option.

1. Let $f(x) = e^x$, $g(x) = \sin^{-1} x$ and $h(x) = f[g(x)]$, then $\frac{h'(x)}{h(x)}$ is equal to

(a) $e^{\sin^{-1}x}$ (b) $\frac{1}{\sqrt{1-x^2}}$

(c) $\sin^{-1}x$ (d) $\frac{1}{(1-x^2)}$

2. If $y = e^{x+e^{x+e^{x+\ldots\text{to}\infty}}}$, find $\frac{dy}{dx}$.

(a) $\frac{y^2}{1-y}$ (b) $\frac{y^2}{y-1}$ (c) $\frac{y}{1-y}$ (d) $\frac{-y}{1-y}$

3. If $y = ae^x + be^{-x} + c$ where a, b, c are parameters, then y' is equal to

(a) $ae^x - be^{-x}$ (b) $ae^x + be^{-x}$

(c) $-(ae^x + be^{-x})$ (d) $ae^x - be^x$

4. The function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ is

(a) continuous at $x = 1$ (b) differentiable at $x = 1$

(c) continuous at $x = 3$ (d) All of these

5. If $y = \sin x + e^x$, then $\frac{d^2x}{dy^2}$ is equal to

(a) $\frac{\sin x - e^x}{(\cos x + e^x)^2}$ (b) $\frac{\sin x - e^x}{(\cos x + e^x)^3}$

(c) $\frac{\sin x - e^x}{(\cos x + e^x)}$ (d) $(-\sin x + e^x)^{-1}$

6. If $y = \tan^{-1}\left(\frac{\sqrt{x} - x}{1 + x^{3/2}}\right)$, then $y'(1)$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) -1 (d) $-\frac{1}{4}$

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7. If $y = x^3 + x^2 + x + 1$, then y
   (a) has a local minimum
   (b) has a local maximum
   (c) neither have a local minimum nor local maximum
   (d) None of these

8. The equation of the normal to the curve $y = \sin x$ at $(0, 0)$ is
   (a) $x = 0$ (b) $x + y = 0$
   (c) $y = 0$ (d) $x - y = 0$

9. The function $f(x) = x + \cos x$ is
   (a) always increasing
   (b) always decreasing
   (c) increasing for certain range of x
   (d) None of these

10. The equation of the tangent to the curve $y = e^{2x}$ at the point $(0, 1)$ is
    (a) $y + 1 = 2x$ (b) $1 - y = 2x$
    (c) $y - 1 = 2x$ (d) None of these

(Q. 11-Q.15) Fill in the blanks.

11. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ _____.

12. If $y = \tan^{-1} x + \cot^{-1} x$, $x \in R$, then $\frac{dy}{dx}$ is equal to _____.

OR

The function $f(x) = \frac{x}{\log x}$ increases in the interval _____.

13. Tangent to the curve $y = x^3 + 3x$ at $x = -1$ and $x = 1$ are _____.

14. If $f(x) = x + 1$, then value of $\frac{d}{dx}(fof)(x)$ is _____.

OR

If $y = e^{3x+7}$, then the value of $\left[\frac{dy}{dx}\right]_{x=0}$ is equal to _____.

15. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals _____.

(Q.16-Q.20) Answer the following questions.

16. If $y = \log(\cos e^x)$, then find $\frac{dy}{dx}$.

17. Let $f(x) = x|x|$, for all $x \in R$ check its differentiability at $x = 0$.

18. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing.

OR

Find the slope of the tangent to the curve $y = 2\sin^2(3x)$ at $x = \frac{\pi}{6}$.

19. Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3} & , \ x \neq 3 \\ k & , \ x = 3 \end{cases}$$

20. Find the least value of the function $f(x) = ax + b/x$. $(a > 0, b > 0, x > 0)$

SECTION - B

21. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

22. If $x = a \sec\theta$, $y = b \tan\theta$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

OR

Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.

23. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$.

24. Find the points on the curve $y = x^3 - 3x^2 - 4x$ at which the tangent lines are parallel to the line $4x + y - 3 = 0$.

OR

Find $\frac{dy}{dx}$ at $x = 1$, $y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

25. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

26. Find the interval in which the function $f(x) = \tan^{-1}(\sin x + \cos x)$ is increasing.

SECTION - C

27. If $x^y - y^x = a^b$, then find $\frac{dy}{dx}$.

28. If $x^m y^n = (x + y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

OR

Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$.

29. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$ with respect to $\cos^{-1} x^2$.

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30. If $y = (x + \sqrt{1+x^2})^n$, then show that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y.$$

31. Find the intervals in which the function

$$f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11 \text{ is}$$

(i) strictly increasing

(ii) strictly decreasing.

OR

Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not.

$$f(x) = \begin{cases} x, & x < 1 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$$

32. For what value of a is the function f defined by

$$f(x) = \begin{cases} a \sin\frac{\pi}{2}(x+1), & x \le 0 \\ \dfrac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

continuous at $x = 0$?

SECTION - D

33. Show that function $f(x) = e^{|x|}$ is continuous everywhere but not differentiable at $x = 0$.

34. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with both the axes. Also, find the equation of the normals.

OR

Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is (i) strictly increasing (ii) strictly decreasing.

35. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing.

OR

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank?

36. Find the equation of tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$.

SOLUTIONS

1. (b) : $f(x) = e^x$ and $g(x) = \sin^{-1}x$ and $h(x) = f[g(x)]$

$\Rightarrow \quad h(x) = f(\sin^{-1}x) = e^{\sin^{-1}x}$

$\therefore \quad h'(x) = e^{\sin^{-1}x}\left(\frac{1}{\sqrt{1-x^2}}\right) = h(x)\cdot\frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \quad \frac{h'(x)}{h(x)} = \frac{1}{\sqrt{1-x^2}}$

2. (c) : We may write the given series as

$y = e^{x+y} \Rightarrow \log y = (x + y)$

Differentiating w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \left(\frac{1}{y} - 1\right)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{(1-y)}{y}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{y}{(1-y)}$$

3. (a) : $y = ae^x + be^{-x} + c$

Differentiating w.r.t. x, we get

$y' = ae^x - be^{-x}$

4. (d) : $|x - 3|$ is continuous at $x = 3$, but not differentiable.

$f(1^-) = f(1^-) = f(1) = 2$

$f'(1^-) = f'(1^+) = -1 = f'(1)$

$\therefore \quad f(x)$ is continuous and differentiable at $x = 1$.

5. (b) : $y = \sin x + e^x$

Differentiating w.r.t., we get

$$\frac{dy}{dx} = \cos x + e^x \Rightarrow \frac{dx}{dy} = \frac{1}{\cos x + e^x}$$

$$\therefore \quad \frac{d^2x}{dy^2} = -\frac{1}{(\cos x + e^x)^2}[-\sin x + e^x]\frac{dx}{dy}$$

$$= -\frac{(e^x - \sin x)}{(\cos x + e^x)^2} \times \frac{1}{\cos x + e^x} = \frac{\sin x - e^x}{(\cos x + e^x)^3}$$

6. (d) : $y = \tan^{-1}\left(\frac{\sqrt{x} - x}{1 + x^{3/2}}\right) = \tan^{-1}\left(\frac{\sqrt{x} - x}{1 + \sqrt{x}\cdot x}\right)$

$$= \tan^{-1}(\sqrt{x}) - \tan^{-1}(x)$$

Differentiating w.r.t. x, we get

$$y' = \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}} - \frac{1}{1+x^2} \Rightarrow y'(1) = \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2} = -\frac{1}{4}$$

7. (c) : Let $f(x) = y = x^3 + x^2 + x + 1$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 + 2x + 1$.

For a maximum or minimum, we have

$f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$

But, this equation gives imaginary values of x.

So, $f'(x) \ne 0$ for any real value of x.

Hence, $f(x)$ does not have a maximum or minimum.

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8. **(b)** : Since, $\frac{dy}{dx} = \cos x$, therefore, slope of tangent at $(0, 0) = \cos 0 = 1$ and hence slope of normal at $(0, 0)$ is -1.

$\therefore$ Equation of normal is $y - 0 = -1(x - 0)$

$\Rightarrow \quad x + y = 0$

9. **(a)** : We have, $f(x) = x + \cos x$

Differentiating w.r.t. x, we get $f'(x) = 1 - \sin x$

Since, $f'(x) > 0$ for all values of x

(As, $\sin x$ is lying between -1 to 1)

$\therefore \quad f(x)$ is always increasing.

10. **(c)** : We have, $y = e^{2x}$

$\left(\frac{dy}{dx}\right)_{(0,1)} = 2e^0 = 2$

Hence, equation of tangent to the given curve at $(0, 1)$ is

$y - 1 = 2(x - 0) \Rightarrow y - 1 = 2x$

11. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x = 1$.

12. We have, $y = \tan^{-1} x + \cot^{-1} x$

$\Rightarrow \quad y = \frac{\pi}{2} \qquad (\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2})$

$\Rightarrow \quad \frac{dy}{dx} = 0$

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OR

$f(x) = \frac{x}{\log x}$ is defined for $x > 0$ and $x \neq 1$

Also, $f'(x) = \frac{\log x \cdot 1 - x \cdot \frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$

$\therefore \quad f'(x) > 0 \Rightarrow \log x > 1 \Rightarrow x > e \quad \therefore \quad x \in (e, \infty)$.

13. Here, $\frac{dy}{dx} = 3x^2 + 3$

$\Rightarrow$ slopes of tangents at $x = 1$ and $x = -1$ are equal.

Hence, the two tangents are parallel.

14. Given, $f(x) = x + 1$

Now, $(fof)(x) = f(f(x)) = f(x + 1)$

$= (x + 1) + 1 = x + 2$

$\therefore \quad \frac{d}{dx}(fof)(x) = \frac{d}{dx}(x + 2) = 1$

OR

We have, $y = e^{3x+7} \Rightarrow \frac{dy}{dx} = 3e^{3x+7}$

$\therefore \quad \left[\frac{dy}{dx}\right]_{x=0} = 3e^{3\times 0+7} = 3e^7$

15. We have, $y = \log_e\left(\frac{x^2}{e^2}\right)$

$\therefore \quad \frac{dy}{dx} = \frac{e^2}{x^2} \cdot \frac{1}{e^2} \cdot 2x = \frac{2}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x^2}$

16. Given $y = \log(\cos e^x)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{1}{\cos e^x}(-\sin e^x \cdot e^x) = -e^x \tan e^x$

17. To check the differentiability of $f(x) = x|x|$ at $x = 0$.

Consider, $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h|h| - 0}{h} = \lim_{h\to 0} |h| = 0$

Hence, $f'(0)$ exists, so $f(x) = x|x|$ is differentiable at $x = 0$.

18. Let $y = f(x) = 7 - 4x - x^2$

$\therefore \quad \frac{dy}{dx} = -4 - 2x$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow \quad -4 - 2x > 0 \Rightarrow x < -2$

$\therefore$ Required interval is $(-\infty, -2)$

OR

We have, $y = 2\sin^2(3x)$ ...(i)

Slope of tangent to (i) is given by

$\frac{dy}{dx} = 4\sin(3x) \cdot \cos(3x) \cdot 3 = 6\sin 6x$

Now, $\left[\frac{dy}{dx}\right]_{x=\pi/6} = 6\sin 6 \cdot \frac{\pi}{6} = 6\sin \pi = 6 \times 0 = 0$

19. Given, $f(x)$ is continuous at $x = 3$.

So, $\lim_{x\to 3} f(x) = f(3) \Rightarrow \lim_{x\to 3} \frac{(x+3)^2 - 36}{x - 3} = k$

$\Rightarrow \lim_{x\to 3} \frac{(x+3)^2 - 6^2}{x - 3} = k$

$\Rightarrow \lim_{x\to 3} \frac{(x+3+6)(x+3-6)}{x - 3} = k$

$\Rightarrow \quad 3 + 3 + 6 = k$

$\Rightarrow \quad k = 12$

Monthly Test Drive-4 CLASS XII ANSWER KEY

1.	(b)	**2.**	(d)	**3.**	(b)	**4.**	(b)	**5.**	(a)
6.	(c)	**7.**	(b, c)	**8.**	(a, b)	**9.**	(b, d)	**10.**	(c, d)
11.	(a, b)	**12.**	(a, c)	**13.**	(b, c)	**14.**	(a)	**15.**	(b)
16.	(d)	**17.**	(2)	**18.**	(4)	**19.**	(2)	**20.**	(7)

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20. As we know that, A.M. $\geq$ G.M.

$\therefore \quad \frac{1}{2}\left(ax+\frac{b}{x}\right) \geq \sqrt{ax.\frac{b}{x}} \Rightarrow \left(ax+\frac{b}{x}\right) \geq 2\sqrt{ab}$

So, least value of $f(x)$ is $2\sqrt{ab}$

21. Let, $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$$

$$= \frac{d}{dx}\tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)$$

$$= \frac{d}{dx}\tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = -\frac{1}{2}$$

22. We have, $x = a\sec\theta$, $y = b\tan\theta$

$\therefore \quad \frac{dx}{d\theta} = a\sec\theta\tan\theta$ and $\frac{dy}{d\theta} = b\sec^2\theta$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$

$$= \frac{b}{a}\cdot\frac{1}{\sin\theta}$$

$\therefore \quad \left[\frac{dy}{dx}\right]_{\theta=\frac{\pi}{3}} = \frac{b}{a}\cdot\frac{1}{\sin\frac{\pi}{3}} = \frac{b}{a}\cdot\frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{a}$

OR

We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$\Rightarrow \quad f'(x) = 12x^2 - 36x + 27$

$$= 12\left(x^2 - 3x + \frac{9}{4} - \frac{9}{4}\right) + 27$$

$$= 12\left(x-\frac{3}{2}\right)^2 - 27 + 27 = 12\left(x-\frac{3}{2}\right)^2 \geq 0 \ \forall\, x \in R$$

Hence, $f(x)$ is always increasing on R.

23. We have, $x = a\cos\theta$, $y = b\sin\theta$

$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$

So, $\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

Now, $\frac{d^2y}{dx^2} = -\frac{b}{a}(-\text{cosec}^2\theta)\frac{d\theta}{dx}$

$$= \frac{b}{a}\text{cosec}^2\theta\left(-\frac{1}{a}\text{cosec}\,\theta\right) = -\frac{b}{a^2}\text{cosec}^3\theta$$

24. The given curve is $y = x^3 - 3x^2 - 4x$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x - 4$$

Since, the tangent is parallel to $4x + y - 3 = 0$

$\therefore \quad \frac{dy}{dx}$ = Slope of line $= -4$

$\Rightarrow \quad 3x^2 - 6x - 4 = -4$

$\Rightarrow \quad 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0, 2$

Putting the value of $x = 0, 2$, in (i), we get

$y = 0$ and $y = -12$

$\therefore \quad$ The points are $(0, 0)$ and $(2, -12)$.

OR

We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x on both sides, we get

$$2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2} - \sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2}-1)}$$

25. Let $u = \sin^2 x$, $v = e^{\cos x}$

$\therefore \quad \frac{du}{dx} = 2\sin x\cos x$ and $\frac{dv}{dx} = e^{\cos x}(-\sin x)$

Now, $\frac{du}{dv} = \frac{du}{dx}\cdot\frac{dx}{dv} = \frac{2\sin x\cos x}{e^{\cos x}(-\sin x)} = \frac{-2\cos x}{e^{\cos x}}$

26. Since, $f(x) = \tan^{-1}(\sin x + \cos x)$

$\therefore \quad f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x - \sin x)$

$$= \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{1+(\sin x+\cos x)^2}$$

$f(x)$ is increasing, if $f'(x) > 0 \Rightarrow \cos\left(x+\frac{\pi}{4}\right) > 0$

$$\Rightarrow \quad -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

27. We have, $x^y - y^x = a^b$

Taking log on both sides, we get

$y\log x - x\log y = b\log a$

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Differentiating w.r.t. x, we get

$$\frac{y}{x}+(\log x)\frac{dy}{dx}-\log y-\frac{x}{y}\cdot\frac{1}{x}\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{y}{x}-\log y\right)=\frac{dy}{dx}\left(\frac{x}{y}-\log x\right)$$

$$\Rightarrow \frac{1}{x}(y-x\log y)=\frac{dy}{dx}\left(\frac{x-y\log x}{y}\right)\Rightarrow \frac{dy}{dx}=\frac{y}{x}\frac{(y-x\log y)}{(x-y\log x)}$$

28. Given, $x^m y^n=(x+y)^{m+n}$

Taking log on both sides, we get

$\log x^m+\log y^n=(m+n)\log(x+y)$

$\Rightarrow \quad m\log x+n\log y=(m+n)\log(x+y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x}+n\cdot\frac{1}{y}\cdot\frac{dy}{dx}=(m+n)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right)=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right)=\frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right)=\frac{nx-my}{x(x+y)} \quad \therefore \frac{dy}{dx}=\frac{y}{x}$$

OR

Refer to answer 35, page no. 124 MTG CBSE Champion, Mathematics Class-XII.

29. Let $u=\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ ...(i)

Putting $x^2=\cos 2\theta$ in (i), we get

$$u=\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$=\tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]=\tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right]=\frac{\pi}{4}-\theta$$

$$=\frac{\pi}{4}-\frac{\cos^{-1}(x^2)}{2}$$

$$\Rightarrow \frac{du}{dx}=-\frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x=\frac{x}{\sqrt{1-x^4}} \quad \text{... (ii)}$$

Let $v=\cos^{-1}(x^2) \Rightarrow \frac{dv}{dx}=\frac{-2x}{\sqrt{1-x^4}}$ ... (iii)

Dividing (ii) by (iii), we get $\frac{du}{dv}=-\frac{1}{2}$

30. We have, $y=(x+\sqrt{1+x^2})^n$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=n\left(x+\sqrt{1+x^2}\right)^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}=\frac{ny}{\sqrt{1+x^2}} \quad \text{...(i)}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\cdot\frac{dy}{dx}-\frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}}-\frac{xy}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-\frac{nxy}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-x\frac{dy}{dx} \quad \text{[From (i)]}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

31. Here, $f(x)=\frac{3}{10}x^4-\frac{4}{5}x^3-3x^2+\frac{36}{5}x+11$

Differentiating w.r.t. x, we get

$$f'(x)=\frac{3}{10}\cdot 4x^3-\frac{4}{5}\cdot 3x^2-3\cdot 2x+\frac{36}{5}\cdot 1$$

$$=\frac{6}{5}(x^3-2x^2-5x+6) \quad =\frac{6}{5}(x-1)(x^2-x-6)$$

$$=\frac{6}{5}(x-1)(x+2)(x-3)$$

$\therefore \quad f'(x)=0 \Rightarrow x=-2, 1, 3.$

Hence, the points divide the real line into disjoint intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -2)$	$(-)\ (-)\ (-) < 0$	Strictly Decreasing
$(-2, 1)$	$(-)\ (+)\ (-) > 0$	Strictly Increasing
$(1, 3)$	$(+)\ (+)\ (-) < 0$	Strictly Decreasing
$(3, \infty)$	$(+)\ (+)\ (+) > 0$	Strictly Increasing

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(i) $f(x)$ is strictly increasing in $(-2, 1) \cup (3, \infty)$
(ii) $f(x)$ is strictly decreasing in $(-\infty, -2) \cup (1, 3)$

OR

Refer to answer 29, page no. 97-98 MTG CBSE Champion, Mathematics Class-XII.

32. $\because f(x)$ is continuous at $x = 0$,

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ ...(1)

Here, $f(0) = a\sin\dfrac{\pi}{2} = a$

Now, $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} a\sin\left(\dfrac{\pi}{2}(-h+1)\right)$

$= \lim_{h\to 0} a\sin\left(\dfrac{\pi}{2} - h\dfrac{\pi}{2}\right) = a$

Also, $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \dfrac{\tan h - \sin h}{h^3}$

$= \lim_{h\to 0} \dfrac{\dfrac{\sin h}{\cos h} - \sin h}{h^3} = \lim_{h\to 0} \dfrac{\sin h\left(\dfrac{1}{\cos h} - 1\right)}{h^3}$

$= \lim_{h\to 0} \dfrac{\sin h}{h} \times \lim_{h\to 0} \dfrac{1-\cos h}{\cos h \cdot h^2}$

$= 1 \times \lim_{h\to 0} \dfrac{1}{\cos h} \times \lim_{x\to 0} \dfrac{2\sin^2\left(\dfrac{h}{2}\right)}{4\times\left(\dfrac{h}{2}\right)^2} = 1\times\dfrac{1}{2} = \dfrac{1}{2}$

$\therefore\ a = \dfrac{1}{2}$ [Using (i)]

Hence, $f(x)$ is continuous at $x = 0$, if $a = \dfrac{1}{2}$.

33. $f(x) = e^{|x|}$; $f(x) = \begin{cases} e^{-x}, & x < 0 \\ e^{x}, & x \geq 0 \end{cases}$

$\therefore\ e^x$ is continuous for all $x > 0$ and e^{-x} is continuous for all $x < 0$.

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-x} = e^0 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x} = e^0 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} f(x) = f(0)$

Hence, $f(x)$ is continuous everywhere.

$Lf'(0) = \lim_{h\to 0^-} \dfrac{f(0-h) - f(0)}{h}$

$= \lim_{h\to 0^-} \dfrac{e^{-h} - 1}{h} = \lim_{h\to 0^-} \dfrac{\left(1 - h + \dfrac{h^2}{2!} - \dfrac{h^3}{3!} + \ldots\right) - 1}{h}$

$= \lim_{h\to 0^-} \left(-1 + \dfrac{h}{2!} - \dfrac{h^2}{3!} + \ldots\right) = -1$

$Rf'(0) = \lim_{h\to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \dfrac{e^h - 1}{h}$

$= \lim_{h\to 0^+} \dfrac{\left(1 + h + \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + \ldots\right) - 1}{h} = \lim_{h\to 0^+} \left(1 + \dfrac{h}{2!} + \dfrac{h^2}{3!} + \ldots\right) = 1$

$Lf'(0) \neq Rf'(0)$

So, $f(x)$ is not differentiable at $x = 0$.

34. Let the required point be (x, y)

The equation of curve is $9y^2 = x^3$

Since (x_1, y_1) lies on the curve

$\therefore\ 9y_1^2 = x_1^3$ ...(i)

Now, $9y^2 = x^3 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^2}{6y}$

$\therefore \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} = \dfrac{x_1^2}{6y_1}$

Since, the normal to the curve at (x_1, y_1) make equal intercept with the coordinate axes.

$\therefore$ Slope of normal $= \pm 1$

$\Rightarrow -\dfrac{1}{\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}} = \pm 1 \Rightarrow \left[\dfrac{dy}{dx}\right]_{x_1, y_1} = \pm 1$

$\Rightarrow \dfrac{x_1^2}{6y_1} = \pm 1 \Rightarrow x_1^4 = 36y_1^2$

$\Rightarrow x_1^4 = 36\left(\dfrac{x_1^3}{9}\right)$ [Using (i)]

$\Rightarrow x_1^4 = 4x_1^3 \Rightarrow x_1^3(x_1 - 4) = 0 \Rightarrow x_1 = 0, 4$

when $x_1 = 0$, from (i) $y_1 = 0$

when $x_1 = 4$, from (i), $y_1 = \pm 8/3$

But the line making equal intercepts with the coordinate axes cannot pass through the origin.

Hence, the required points are

$\left(4, \dfrac{8}{3}\right)$ and $\left(4, -\dfrac{8}{3}\right)$

The equations of normal is $y - \dfrac{8}{3} = -1(x - 4)$

$\Rightarrow\ y + x = 4 + \dfrac{8}{3} \Rightarrow y + x = \dfrac{20}{3}$

And $y + \dfrac{8}{3} = 1(x - 4)$

$\Rightarrow\ y - x = -4 - \dfrac{8}{3} \Rightarrow y - x = \dfrac{-20}{3}$

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OR

Refer to answer 30, page no. 123 MTG CBSE Champion, Mathematics Class-XII.

35. $f(x) = \sin 3x - \cos 3x \Rightarrow f'(x) = 3\cos 3x + 3\sin 3x$

$f'(x) = 0 \Rightarrow 3\cos 3x = -3\sin 3x$

$\Rightarrow \quad \cos 3x = -\sin 3x \Rightarrow \tan 3x = -1,$

which gives $3x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{11\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$ $\quad [\because 0 < x < \pi]$

The points $x = \frac{\pi}{4}, x = \frac{7\pi}{12}$ and $x = \frac{11\pi}{12}$ divides the interval $(0, \pi)$ into four disjoint intervals, $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$

Now, $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$

$\Rightarrow$ f is strictly increasing in $\left(0, \frac{\pi}{4}\right)$

$f'(x) < 0$ in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$f'(x) > 0$ in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$\Rightarrow$ f is strictly increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$f'(x) < 0$ in $\left(\frac{11\pi}{12}, \pi\right)$

$\Rightarrow$ f is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$

Hence, f is strictly increasing in the intervals

$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

and f is strictly decreasing in the intervals

$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$

OR

Refer to answer 72, page no. 132 MTG CBSE Champion, Mathematics Class-XII.

36. Let the point of contact of one of the tangents be (x_1, y_1).

Then (x_1, y_1) lies on $y = \cos(x + y)$

$\therefore \quad y_1 = \cos(x_1 + y_1)$ ...(i)

Since, the tangents are parallel to the line $x + 2y = 0$.

$\therefore$ Slope of tangent at (x_1, y_1) = slope of line $x + 2y = 0$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{1}{2}$

Since, the equation of curve is $y = \cos(x + y)$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\sin(x_1 + y_1)\left\{1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right\}$

$\Rightarrow \quad -\frac{1}{2} = -\sin(x_1 + y_1)\left(1 - \frac{1}{2}\right)$

$\Rightarrow \quad \sin(x_1 + y_1) = 1$ ...(ii)

Squaring (i) and (ii), then adding, we get

$\cos^2(x_1 + y_1) + \sin^2(x_1 + y_1) = y_1^2 + 1$

$\Rightarrow \quad y_1^2 + 1 = 1 \Rightarrow y_1 = 0$

Put $y_1 = 0$ in (i) and (ii),

$\cos x_1 = 0$ and $\sin x_1 = 1$

$\Rightarrow \quad x_1 = \frac{\pi}{2}, -\frac{3\pi}{2}$

Hence, the points of contact are

$\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$

Since, the slope of the tangent is $-\frac{1}{2}$.

$\therefore$ Equation of tangents at $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$ are

$y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$ and $y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$

or $\quad 2x + 4y - \pi = 0$ and $2x + 4y + 3\pi = 0$

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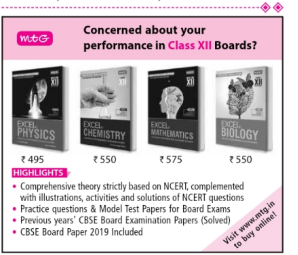

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MONTHLY TEST DRIVE

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80 **Series 4 : Continuity and Differentiability** **Time Taken : 60 Min.**

Only One Option Correct Type

1. Let $f : R \to R$ be a function satisfying $f(x + y) = f(x) + \lambda xy + 3x^2y^2$ for all $x, y \in R$. If $f(3) = 4$ and $f(5) = 52$, then $f'(x)$ is equal to
(a) $10x$ (b) $-10x$ (c) $20x$ (d) $128x$

2. If $x = \sin t - \operatorname{cosec} t$, $y = \sin^5 t - \operatorname{cosec}^5 t$, then $(x^2 + 4)y'' =$
(a) $xy' - 5y$ (b) $-xy' + 5y$
(c) $xy' - 25y$ (d) $-xy' + 25y$

3. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then
(a) g is not differentiable at $x = 0$
(b) $g'(0) = \cos(\log 2)$
(c) $g'(0) = -\cos(\log 2)$
(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

4. If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - \pi/2}, & x \neq \dfrac{\pi}{2} \\ 1, & x = \pi/2 \end{cases}$ where $\{\cdot\}$ represents the fractional part function, then $f(x)$ is
(a) continuous at $x = \pi/2$
(b) $\lim\limits_{x \to \pi/2} f(x)$ exists, $f(x)$ is not continuous at $x = \pi/2$
(c) $\lim\limits_{x \to \pi/2^+} f(x)$ does not exist
(d) none of these

5. The derivative of $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & x \neq 1 \\ -\dfrac{1}{3}, & x = 1 \end{cases}$ at $x = 1$ is
(a) $-\dfrac{2}{9}$ (b) $-\dfrac{2}{44}$
(c) 0 (d) none of these

6. If $f(x) = \left[\dfrac{(x-2)^3}{a}\right] \sin(x - 2) + a \cos(x - 2)$, where $[\cdot]$ denotes the greatest integer function, is continuous and differentiable in $(4, 6)$, then
(a) $a \in [8, 64]$ (b) $a \in (0, 8]$
(c) $a \in [64, \infty)$ (d) $a \in (0, 6) \cup (6, 10]$

One or More Than One Option(s) Correct Type

7. If $f(x) = \begin{cases} \dfrac{x}{1 + e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then
(a) $f'(0^+) = 1$ (b) $f'(0^+) = 0$
(c) $f'(0^-) = 1$ (d) $f'(0^-) = 0$

8. If $f(x) = \begin{cases} \dfrac{x}{1 + |x|}, & |x| \geq 1 \\ \dfrac{x}{1 - |x|}, & |x| < 1 \end{cases}$, then which of the following is false?
(a) $f(x)$ is discontinuous and non-differentiable at $x = -1, 1, 0$
(b) $f(x)$ is discontinuous and non-differentiable at $x = -1$, whereas continuous and differentiable at $x = 0, 1$
(c) $f(x)$ is discontinuous and non-differentiable at $x = -1, 1$, whereas continuous and differentiable at $x = 0$.
(d) none of these

9. If the function $f(x) = \begin{cases} x + a^2\sqrt{2}\sin x, & 0 \leq x < \pi/4 \\ x \cot x + b, & \pi/4 \leq x < \pi/2 \\ b \sin 2x - a \cos 2x, & \pi/2 \leq x \leq \pi \end{cases}$ is continuous in the interval $[0, \pi]$, then the ordered pair (a, b) is
(a) $(-1, -1)$ (b) $(0, 0)$
(c) $(-1, 1)$ (d) $(1, 1)$

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10. If $f(x) = \frac{x}{2} - 1$, then on the interval $[0, \pi]$

(a) $\tan f(x)$ and $\frac{1}{f(x)}$ are continuous

(b) $\tan f(x)$ and $\frac{1}{f(x)}$ are discontinuous

(c) $\tan f(x)$ and $\frac{1}{f^{-1}(x)}$ are continuous

(d) $\tan f(x)$ is continuous but $\frac{1}{f(x)}$ is not continuous

11. If $f(x) = x\left(\sqrt{x} + \sqrt{x+1}\right)$, then which of the following statements is (are) false?

(a) $f(x)$ is not continuous at $x = 0$

(b) $f(x)$ is continuous at $x = 0$, but not differentiable at $x = 0$

(c) $f(x)$ is differentiable at $x = 0$

(d) none of these

12. Identify the false statement for

$$f(x) = \begin{cases} \frac{x^2}{2} & , \quad 0 \le x < 1 \\ 2x^2 - 3x + \frac{3}{2} & , \quad 1 \le x \le 2 \end{cases}$$

(a) f, f' and f'' are continuous in $[0, 2]$

(b) f and f' are continuous in $[0, 2]$ whereas f'' is continuous in $[0, 1) \cup (1, 2]$

(c) f and f' are differentiable in $[0, 2]$

(d) none of these

13. If $f(x) = \cos\left[\frac{\pi}{x}\right]\cos\left(\frac{\pi}{2}(x-1)\right)$, where $[\cdot]$ denotes the greatest integer function, then $f(x)$ is continuous at

(a) $x = 0$ (b) $x = 1$

(c) $x = 2$ (d) none of these

Comprehension Type

If $f(x + y) = f(x) + f(y)$ for all $x, y \in R$ and $f(1) = 1$ and $g(0) = \lim_{x \to 0} \frac{2^{f(\tan x)} - 2^{f(\sin x)}}{x^2 f(\sin x)}$. then,

14. The value of $f(x)$ is

(a) x (b) x^2

(c) $3x$ (d) none of these

15. The value of $g(0)$ is

(a) $\log_e 2$ (b) $\frac{1}{2}\log_e 2$

(c) $2\log_e 2$ (d) $\log_e\left(\frac{1}{2}\right)$

Matrix Match Type

16. Match the following:

	Column-I		Column-II
P.	If $f(x) = \begin{cases} x, & x \text{ is rational} \\ x^2, & x \text{ is irrational} \end{cases}$ then the number of points at which $f(x)$ is continuous is	1.	0
Q.	The number of solutions of the equation $[x-1] + [x+1] = 0$ is	2.	1
R.	If $f(x) = x^2 \sin\left(\frac{1}{x}\right), x \neq 0$, $f(0) = 0$, then $f'(0^-)$ is	3.	2
S.	If $f(x) = \lvert x-1\rvert + \lvert x\rvert + \lvert x+1\rvert$, then $f'(0^+)$ is	4.	> 3

	P	Q	R	S
(a)	1	3	4	2
(b)	2	4	3	1
(c)	4	2	3	1
(d)	3	4	1	2

Integer Answer Type

17. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is _____ .

18. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ and $kx\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$, then the value of constant k must be _____ .

19. If $f(x) = \text{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of $|a|$ equals _____ .

20. If Rolle's theorem holds for the function $f(x) = 2x^3 + bx^2 + cx$, $x \in [-1, 1]$, at the point $x = 1/2$, then $2b - 3c$ equals _____ .

◈◈

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SELF CHECK

No. of questions attempted .....
No. of questions correct .....
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Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

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BRAIN @ WORK

ELLIPSE

This article is a collection of shortcut methods, important formulas and MCQs along with their detailed solutions which provides an extra edge to the readers who are preparing for various competitive exams like JEE(Main & Advanced) and other PETs.

DEFINITION AND STANDARD EQUATION OF ELLIPSE

An ellipse is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is less than unity.

The constant ratio is generally denoted by e $(0 < e < 1)$ and is known as the eccentricity of the ellipse.

The general equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 < ab$.

EQUATION OF THE ELLIPSE

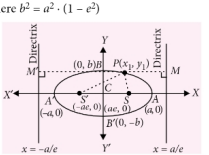
- The equation of the ellipse, whose focus is the point (h, k) and directrix is $lx + my + n = 0$ and having eccentricity e, is

$$(x-h)^2 + (y-k)^2 = e^2 \cdot \frac{(lh+mk+n)^2}{(l^2+m^2)}$$

- Equation of the ellipse whose focus is $(ae, 0)$ and directrix is $x = a/e$ and having eccentricity e is given by

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1 \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2 \cdot (1 - e^2)$



The ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(1)

The foci S and S' are $(ae, 0)$ and $(-ae, 0)$.

The equation of its directrices are $x = a/e$ and $x = -a/e$.

Let $P(x_1, y_1)$ be any point on (1),

Now, $SP = ePM = e\,(a/e - x_1) = a - ex_1$

and $S'P = ePM' = e\,(a/e + x_1) = a + ex_1$

$\therefore \quad SP + S'P = (a - ex_1) + (a + ex_1) = 2a = AA'$

= Length of major axis

The sum of focal distances of any point on the ellipse is equal to the length of major axis.

Note :

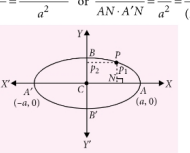
(a) Two ellipses are said to be similar, if they have the same value of eccentricity.

(b) Distance of every focus from the extremity of minor axis is equal to a, as $b^2 + a^2e^2 = a^2$.

- Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can also be defined as

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} \quad \Rightarrow \quad \frac{y^2}{b^2} = \frac{(a+x)(a-x)}{a^2}$$

$$\text{or} \quad \frac{(PN)^2}{b^2} = \frac{A'N \cdot AN}{a^2} \quad \text{or} \quad \frac{(PN)^2}{AN \cdot A'N} = \frac{b^2}{a^2} = \frac{(BC)^2}{(AC)^2}$$



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- **Equation of Ellipse Referred to Two Perpendicular Lines**

(a) Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can also be defined as

$$\frac{(\text{L.P. from any point on ellipse to the major axis})^2}{(\text{Length of semi minor axis})^2}$$
$$+\frac{(\text{L.P. from any point on ellipse to the minor axis})^2}{(\text{Length of semi major axis})^2}=1$$

(where L.P. means length of perpendicular)

$$\Rightarrow \frac{p_1^2}{b^2}+\frac{p_2^2}{a^2}=1$$

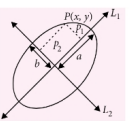
(b) If $L_1 : a_1x+b_1y+c_1=0$ and $L_2 : b_1x-a_1y+c_2=0$, then the equation

$$\frac{\left(\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}\right)^2}{b^2}+\frac{\left(\frac{b_1x-a_1y+c_2}{\sqrt{a_1^2+b_1^2}}\right)^2}{a^2}=1$$

represents an ellipse in the plane such that

(i) The centre of the ellipse is the point of intersection of the lines $L_1=0$ and $L_2=0$.

(ii) If $a > b$, then the major axis lies along $L_2=0$ and the minor axis along $L_1=0$.
If $a < b$, then the major axis is along $L_1=0$ and minor axis is along $L_2=0$.



(iii) If $a > b$, then the lengths of the major and minor axes are $2a$ and $2b$ respectively and if $a < b$, then the lengths of major and minor axes are $2b$ and $2a$ respectively.

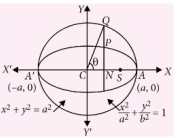
- **Parametric Equation of the Ellipse**

The circle described on the major axis of an ellipse as diameter is called the auxiliary circle of the ellipse.

Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$.

$\therefore$ Equation of its auxiliary circle is $x^2+y^2=a^2$
($\because$ AA' is diameter of the circle)

Let Q be a point on auxiliary circle $x^2+y^2=a^2$ such that QP produced is perpendicular to the x-axis.
Then P and Q are the corresponding points on the ellipse and the auxiliary circle respectively.



Let $\angle QCA=\theta \quad (0\le\theta<2\pi)$
i.e., the eccentric angle of P on an ellipse is the angle which the radius (or radius vector) through the corresponding point on the auxiliary circle makes with the major axis.
$\therefore \quad Q\equiv(a\cos\theta,\ a\sin\theta)$ and $P\equiv(a\cos\theta\ ,\ b\sin\theta)$
The equation $x=a\cos\theta$ and $y=b\sin\theta$ taken together are called parametric equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is real parameter and $P\equiv(a\cos\theta\ ,\ b\sin\theta\ )$ is any point on the ellipse also known as $P(\theta)$ or 'θ' point on the ellipse.

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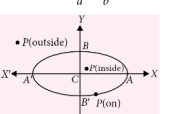
Note:

(i) Eccentric angles of the extremities of latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.

(ii) Area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab sq. units.

Position of Points w.r.t. an Ellipse

Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $P\equiv(h,\ k)$ be any point, then P will lie outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as $\frac{h^2}{a^2}+\frac{k^2}{b^2}>,=,<1$



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Important Points

The equation of the ellipse whose axes are parallel to the coordinate axes and whose centre is at the origin, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with the following properties :

S. No.	Ellipse / Basic Fundamentals	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
(i)	Coordinates of the centre	(0, 0)	(0, 0)
(ii)	Coordinates of the vertices	$(\pm a, 0)$	$(0, \pm b)$
(iii)	Coordinates of the foci	$(\pm ae, 0)$	$(0, \pm be)$
(iv)	Length of the major axis	$2a$	$2b$
(v)	Length of the minor axis	$2b$	$2a$
(vi)	Equation of the major axis	$y=0$	$x=0$
(vii)	Equation of the minor axis	$x=0$	$y=0$
(viii)	Equations of the directrices	$x=\pm a/e$	$y=\pm b/e$
(ix)	Eccentricity	$e=\sqrt{1-\left(\frac{b^2}{a^2}\right)}$	$e=\sqrt{1-\left(\frac{a^2}{a^2}\right)}$
(x)	Ends of the latus rectum	$(\pm ae, \pm b^2/a)$	$(\pm a^2/b, \pm be)$
(xi)	Length of the latus rectum	$2b^2/a$	$2a^2/b$
(xii)	Parametric coordinates	$(a\cos\theta, b\sin\theta)$	$(a\cos\theta, b\sin\theta)$
(xiii)	Auxilliary circle	$x^2+y^2=a^2$	$x^2+y^2=b^2$

Equation of Chord

- Equation of the chord joining the points whose eccentric angles are α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is given by

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right).$$

- If the chord joining two points whose eccentric angles are α and β, cut the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ at the point $(ae, 0)$, then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}.$$

- If α and β are the eccentric angles of extremities of a focal chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$, then

$$e=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \Rightarrow \tan\frac{\alpha}{2}\cdot\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

- Locus of mid-points of focal chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm\frac{ex}{a}$.

Equation of Tangents in Different Forms

Point Form : Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

Parametric Form : Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

Slope Form : Equation of tangent of slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are given by $y=mx\pm\sqrt{(a^2m^2+b^2)}$ and the coordinates of the points of contact are

$$\left(\mp\frac{a^2m}{\sqrt{(a^2m^2+b^2)}}, \pm\frac{b^2}{\sqrt{(a^2m^2+b^2)}}\right).$$

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Results Related to Tangents

(i) The equations of the tangents to the ellipse at points $P(a\cos\theta_1, b\sin\theta_1)$ and $Q(a\cos\theta_2, b\sin\theta_2)$ are $\frac{x}{a}\cos\theta_1 + \frac{y}{b}\sin\theta_1 = 1$ and $\frac{x}{a}\cos\theta_2 + \frac{y}{b}\sin\theta_2 = 1$ and these two intersect at the point $\left(\frac{a\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}, \frac{b\sin\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}\right)$.

(ii) The line joining two points on an ellipse, the difference of whose eccentric angle is constant, touches an other ellipse of same eccentricity.

(iii) The locus of the point of intersection of tangents to an ellipse at the points whose eccentric angles differs by constant is an ellipse of the same eccentricity. If the eccentric angles differs by a right angle, then the locus is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$.

(iv) The locus of the point of intersection of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points the sum of whose eccentric angles is constant is a straight line passing through the centre of the ellipse.

(v) If the tangent at P on an ellipse meets the directrix in F, then PF will subtend a right angle at the corresponding focus *i.e.*, $\angle PSF = \pi/2$.

(vi) If SM and $S'M'$ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \cdot S'M' = b^2$ and the point M & M' lie on the auxiliary circle (where $a > b$).

(vii) The tangents drawn from any point on the director circle of a given ellipse to the ellipse are always at right angle.

Equations of Normals in Different Forms

Point Form : Equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$.

Parametric Form : Equation of normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos\theta, b\sin\theta)$ is $ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$.

Slope Form : Equation of normal of slope m to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by $y = mx \mp \frac{m(a^2-b^2)}{\sqrt{(a^2+b^2m^2)}}$ at the points $\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{b^2m}{\sqrt{a^2+b^2m^2}}\right)$

Results Related to Normals

(i) If the line $lx + my + n = 0$ be a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$

(ii) Locus of mid-points of normal chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2} + \frac{b^6}{y^2}\right) = (a^2-b^2)^2$

(iii) Four normals can be drawn from a point to an ellipse.

Intersection of a Circle and an Ellipse

(i) The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can intersect each other at maximum of four points.

(ii) The maximum number of common chord of a circle and an ellipse is six.

(iii) The maximum number of common tangent of a circle and an ellipse is four.

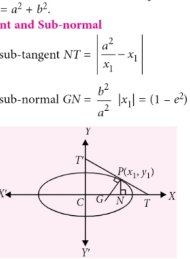
Director Circle

The locus of the point of intersection of the tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are perpendicular to each other is called director circle and its equation is given by $x^2 + y^2 = a^2 + b^2$.

Sub-tangent and Sub-normal

Length of sub-tangent $NT = \left|\frac{a^2}{x_1} - x_1\right|$

Length of sub-normal $GN = \frac{b^2}{a^2}|x_1| = (1 - e^2)|x_1|$



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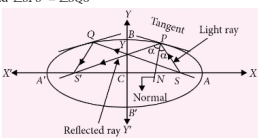
Reflection Property of an Ellipse

If an incoming light ray passes through the focus (S) strike the concave side of the ellipse, then it will get reflected towards other focus (S') and $\angle SPS' = \angle SQS'$



The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.

PROBLEMS

1. The line $y = mx + c$, will be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ($|b| > |a|$) , provided

(a) $c^2 = a^2 + b^2m^2$ (b) $c^2 = b^2 + a^2m^2$

(c) $c^2 = (a^2 + b^2)m^2$ (d) $c^2 = (b^2 - a^2)m^2$

2. Tangents are drawn to the ellipse $x^2 + 2y^2 = 4$ from any arbitrary point on the line $x + y = 4$, the corresponding chord of contact will always pass through a fixed point, whose coordinates are

(a) $\left(1, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 1\right)$

(c) $\left(1, -\frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, -1\right)$

3. Equation of the ellipse whose axes are along the coordinate axes and whose length of latus rectum and eccentricity are equal and equal to 1/2 each , is

(a) $6x^2 + 12y^2 = 1$ (b) $12x^2 + 6y^2 = 1$

(c) $3x^2 + 12y^2 = 1$ (d) $9x^2 + 12y^2 = 1$

4. The line $y = x - 1$ touches the ellipse $3x^2 + 4y^2 = 12$, at

(a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ (b) (3, 2)

(c) (−1, −2) (d) None of these

5. One foot of normal of the ellipse $4x^2 + 9y^2 = 36$, that is parallel to the line $2x + y = 3$, is

(a) $\left(\frac{9}{5}, \frac{8}{5}\right)$ (b) $\left(-\frac{9}{5}, \frac{8}{5}\right)$

(c) $\left(-\frac{9}{5}, -\frac{8}{5}\right)$ (d) None of these

6. Tangent drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point 'P' meets the coordinate axes at points A and B respectively. Locus of mid-point of segment AB is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$

(c) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$ (d) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$

7. Normals drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point 'P' meets the coordinate axes at points A and B respectively. Locus of mid point of segment AB is

(a) $4x^2a^2 + 4y^2b^2 = (a^2 - b^2)^2$

(b) $4x^2b^2 + 4y^2a^2 = (a^2 - b^2)$

(c) $16x^2a^2 + 16y^2b^2 = (a^2 - b^2)^2$

(d) $16x^2b^2 + 16y^2a^2 = (a^2 - b^2)$

8. If the line $y = mx + c$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then corresponding point of contact is

(a) $\left(\frac{a^2m}{c}, \frac{b^2}{c}\right)$ (b) $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$

(c) $\left(\frac{a^2m}{c}, -\frac{b^2}{c}\right)$ (d) $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

9. If the line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then corresponding foot of normal is

(a) $\left(\frac{a^2c}{m(b^2-a^2)}, \frac{b^2c}{(b^2-a^2)}\right)$

(b) $\left(\frac{a^2c}{m(a^2-b^2)}, \frac{b^2c}{(a^2-b^2)}\right)$

(c) $\left(\frac{a^2m}{c(b^2-a^2)}, \frac{b^2c}{(b^2-a^2)}\right)$

(d) $\left(\frac{a^2m}{c(a^2-b^2)}, \frac{b^2c}{(a^2-b^2)}\right)$

10. Locus of the mid-point of chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that are parallel to the line $y = 2x + c$, is

(a) $2b^2y - a^2x = 0$ (b) $2a^2y - b^2x = 0$

(c) $2b^2y + a^2x = 0$ (d) $2a^2y + b^2x = 0$

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11. The locus of the moving point $P(x, y)$ satisfying $\sqrt{(x-1)^2+y^2}+\sqrt{(x+1)^2+(y-\sqrt{12})^2}=a$ will be an ellipse if
(a) $a<4$ (b) $a>2$ (c) $a>4$ (d) $a<2$

12. The equation $\frac{x^2}{6-a}+\frac{y^2}{a-2}=1$ will represent an ellipse if
(a) $a \in (1, 3)$ (b) $a \in (1, 6)$
(c) $a \in (-\infty, 2) \cup (6, \infty)$ (d) $a \in (2, 6) - \{4\}$

13. Foci of the ellipse $16x^2 + 25y^2 = 400$ are
(a) $(\pm 2, 0)$ (b) $(\pm 3, 0)$
(c) $\left(\pm\frac{24}{5}, 0\right)$ (d) $\left(\pm\frac{48}{5}, 0\right)$

14. Equation of an ellipse whose focus is $S \equiv (1, 0)$, corresponding directrix being the line $x + y = 2$ and eccentricity being $\frac{1}{\sqrt{2}}$ is
(a) $3x^2 + 3y^2 - 2xy + 4x - 4y = 0$
(b) $3x^2 + 3y^2 + 2xy - 4x + 4y = 0$
(c) $3x^2 + 3y^2 + 2xy + 4x - 4y = 0$
(d) $3x^2 + 3y^2 - 2xy - 4x + 4y = 0$

15. The equation $3(x + y - 5)^2 + 2(x - y + 7)^2 = 6$ represents an ellipse, whose centre is
(a) $(-1, 6)$ (b) $(6, -1)$ (c) $(1, -6)$ (d) $(-6, 1)$

16. Consider an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, centered at point '$O$' and having AB and CD as it's major and minor axes respectively. If S_1 be one of the focus of the ellipse, radius of incircle of triangle OCS_1 be 1 unit and $OS_1 = 6$ units, then area of the ellipse is equal to
(a) 16π sq. units (b) $\frac{65}{4}\pi$ sq. units
(c) $\frac{65}{2}\pi$ sq. units (d) 65π sq. units

17. P is any variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having the points S_1 and S_2 as its foci. Then maximum area of the triangle PS_1S_2 is equal to
(a) b^2e sq. units (b) a^2e sq. units
(c) ab sq. units (d) abe sq. units

18. P is any variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having the points S_1 and S_2 as its foci. Then Locus of incentre of triangle PS_1S_2 will be
(a) a straight line (b) a circle
(c) a parabola (d) an ellipse

19. P_1 and P_2 are the foot of altitudes drawn from the foci S_1 and S_2 respectively of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ on one of its variable tangent. Maximum value of $(S_1P_1)(S_2P_2)$ is equal to
(a) b^2 (b) b (c) a^2 (d) a

20. The equation of the ellipse centered at (1, 2) having the point (6, 2) as one of its focus and passing through the point (4, 6) is
(a) $\frac{(x-1)^2}{36}+\frac{3(y-2)^2}{64}=1$
(b) $\frac{(x-1)^2}{18}+\frac{(y-2)^2}{32}=1$
(c) $\frac{(x-1)^2}{72}+\frac{7(y-2)^2}{128}=1$
(d) $\frac{(x-1)^2}{45}+\frac{(y-2)^2}{20}=1$

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21. The angle between the tangents drawn to the ellipse $3x^2 + 2y^2 = 5$, from the point $P(1, 2)$ is equal to
(a) $\tan^{-1}(24\sqrt{5})$ (b) $\tan^{-1}(12\sqrt{5})$
(c) $\tan^{-1}\left(\frac{24}{\sqrt{5}}\right)$ (d) $\tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$

22. The line $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If '$\theta$' be the eccentric angle of the foot of this normal, then 'θ' is equal to
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) None of these

23. The distance of a point P (lying in the first quadrant) on the ellipse $x^2 + 3y^2 = 6$ from the center of the ellipse is 2 units. Eccentric angle of the point 'P' is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) None of these

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24. The tangent drawn to the ellipse $\frac{x^2}{16}+\frac{11y^2}{256}=1$ at the point $P(\theta)$, touches the circle $(x-1)^2+y^2=16$. 'θ' is equal to

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) None of these

25. There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the center of the ellipse are equal and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Eccentricity of this ellipse is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{\frac{2}{3}}$

26. The tangent drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at the point '$P$' meets the circle $x^2+y^2=a^2$, $(a>b)$ at the points A and B. The line segment AB subtends an angle $\pi/2$ at the center of the ellipse. If 'e' be the eccentricity of the ellipse and 'θ' be the eccentric angle of point 'P' then

(a) $e^2(1+\sin^2\theta)=1$ (b) $e^2(1+\cos^2\theta)=1$

(c) $e^2(2+\sin^2\theta)=1$ (d) $e^2(2+\cos^2\theta)=1$

27. The line $y=mx+c$, will be a normal to the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if

(a) $\frac{a^2}{m^2}+b^2=\frac{(b^2-a^2)^2}{c^2}$ (b) $\frac{c^2}{m^2}+b^2=\frac{(b^2-a^2)^2}{c^2}$

(c) $a^2+\frac{b^2}{m^2}=\frac{(b^2-a^2)^2}{c^2}$ (d) $c^2+\frac{b^2}{m^2}=\frac{(b^2-a^2)^2}{a^2}$

28. Locus of the point of intersection of tangents drawn to the given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, at points $P(\theta_1)$ and $Q(\theta_2)$, such that $\theta_1-\theta_2=\frac{2\pi}{3}$, is

(a) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=4$ (b) $\frac{a^2}{x^2}+\frac{b^2}{y^2}=2$

(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ (d) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2$

29. Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, having it's eccentricity equal to e. P is any variable point on it and P_1, P_2 are the foot of perpendiculars drawn from P to the x and y-axis respectively. The line P_1P_2 will always be a normal to an ellipse whose eccentricity is equal to

(a) e^2 (b) $\sqrt{e}$ (c) $\sqrt{\frac{2e}{1+e}}$ (d) e

30. The normal drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ at the extremity of the latus rectum passes through the extremity of the minor axis. Eccentricity of this ellipse is equal to

(a) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (b) $\frac{\sqrt{5}-1}{2}$

(c) $\sqrt{\frac{\sqrt{3}-1}{2}}$ (d) $\frac{\sqrt{3}-1}{2}$

31. The equation of common tangents of the curves $x^2-4y^2=8$ and $y^2=4x$ are

(a) $y-2x-4=0$, $y+2x+4=0$

(b) $y-2x-2=0$, $y+2x+2=0$

(c) $2y-x-4=0$, $2y+x+4=0$

(d) None of these

32. The maximum distance of the centre of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ from the chord of contact of mutually perpendicular tangents of the ellipse is

(a) $\frac{9}{\sqrt{13}}$ (b) $\frac{3}{\sqrt{13}}$ (c) $\frac{6}{\sqrt{13}}$ (d) $\frac{36}{\sqrt{13}}$

33. Tangents PA and PB are drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ from the point $P(0, 5)$. Area of triangle PAB is equal to

(a) $\frac{16}{5}$ sq. units (b) $\frac{256}{25}$ sq. units

(c) $\frac{32}{5}$ sq. units (d) $\frac{1024}{25}$ sq. units

34. The straight line $x-2y+4=0$ is one of the common tangents of the parabola $y^2=4x$ and $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. The equation of another common tangent of these curves is

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(a) $x + 2y + 4 = 0$ (b) $x + 2y - 4 = 0$
(c) $x + 2y + 2 = 0$ (d) $x + 2y - 2 = 0$

35. Tangent drawn to the ellipse $\frac{x^2}{36}+\frac{y^2}{9}=1$ at the point 'P' meets the y-axis at A and normal drawn to the ellipse at point 'P' meets the x-axis at B. If area of triangle OAB is $\frac{27}{4}$ sq. units, then eccentric angle of point 'P' is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) None of these

36. The tangent and normal drawn to the ellipse $x^2 + 4y^2 = 4$ at the point P, meet the x-axis at A and B respectively. If $AB = 2$ then cosθ is equal to, ('θ' being the eccentric angle of point P)

(a) $\frac{1}{3}$ (b) $\frac{\sqrt{8}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{\sqrt{5}}{3}$

37. The chord joining the points $P_1(a\ \cos\theta_1,\ b\ \sin\theta_1)$ and $P_2(a\ \cos\theta_2,\ b\ \sin\theta_2)$ meets the x-axis at A. If $OA = c$('O' being the origin), then the value of $\tan\frac{\theta_1}{2}.\tan\frac{\theta_2}{2}$ is equal to

(a) $\frac{c-a}{c+a}$ (b) $\frac{c-a}{c+b}$ (c) $\frac{c-b}{c+b}$ (d) $\frac{c-b}{c+a}$

38. A tangent to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$, having slope $-\frac{4}{3}$ cuts the x and y-axis at the points A and B respectively. If O is the origin then area of triangle OAB is equal to

(a) 12 sq. units (b) 24 sq. units
(c) 36 sq. units (d) None of these

39. Which of the following is a common tangent to the ellipses $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$?

(a) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(b) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(c) $by = ax - \sqrt{a^4 - a^2b^2 + b^4}$

(d) $ay = bx - \sqrt{a^4 - a^2b^2 + b^4}$

40. Maximum length of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, such that eccentric angles of its extremities differ by $\frac{\pi}{2}$ is $(a > b)$

(a) $a\sqrt{2}$ (b) $b\sqrt{2}$
(c) $ab\sqrt{2}$ (d) None of these

41. Consider an ellipse having its axes along the coordinate axes and passing through the point (4, –1). If the line $x + 4y - 10 = 0$ is one of its tangent, then area of the ellipse is equal to

(a) 20 π sq. units (b) 30 π sq. units
(c) 15 π sq. units (d) 10 π sq. units

42. S_1 and S_2 are the foci of an ellipse. 'B' be one of the extremity of its minor axis. If triangle S_1S_2B is right angled, then eccentricity of the ellipse is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{\sqrt{3}}{\sqrt{2}}$ (d) None of these

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43. If the chord of contact of tangents drawn from a point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circles $x^2 + y^2 = c^2$, then locus of P is

(a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^2}{c^2}$ (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{b^4}{c^4}$

(c) $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$ (d) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{a^4}{c^4}$

44. The point on the ellipse $4x^2 + 9y^2 = 1$ such that tangent drawn to the ellipse at this point is perpendicular to the line $8x - 9y = 0$, is

(a) $\left(\frac{27}{2\sqrt{985}},-\frac{16}{3\sqrt{985}}\right)$ (b) $\left(-\frac{27}{2\sqrt{985}},\frac{16}{2\sqrt{985}}\right)$

(c) $\left(\frac{27}{2\sqrt{985}},\frac{16}{3\sqrt{985}}\right)$ (d) None of these

45. Tangents drawn to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, from the point '$P$', meets the coordinate axes at concyclic points, then locus of point 'P' is

(a) $x^2 + y^2 = 7$ (b) $x^2 - y^2 = 7$
(c) $x^2 + y^2 = 25$ (d) $x^2 - y^2 = 25$

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SOLUTIONS

1. (b): If the point of contact is $P(a\cos\theta, b\sin\theta)$, then equation of tangent $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ and given line $y = mx + c$ will be identical. Comparing the coefficients, we get

$$\frac{\cos\theta}{am} = \frac{\sin\theta}{-b} = -\frac{1}{c}$$

$$\Rightarrow \cos\theta = -\frac{am}{c}, \sin\theta = \frac{b}{c} \Rightarrow a^2m^2 + b^2 = c^2$$

2. (a): Let the arbitrary point be $(a, 4 - a)$, then equation of corresponding chord of contact is,

$$x \cdot a + 2y\,(4 - a) = 4$$

$$\Rightarrow a(x - 2y) + 8y - 4 = 0$$

It is indeed a family of concurrent lines. The point of concurrency being the intersection point of the lines $x - 2y = 0$ and $8y - 4 = 0$

$\therefore$ Required point is $\left(1, \frac{1}{2}\right)$

3. (d): $\frac{2b^2}{a} = \frac{1}{2} \Rightarrow \frac{b^2}{a} = \frac{1}{4}$ ...(i) (Given)

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{2} \quad \text{...(ii)}$$

From (i) and (ii), we get, $a = \frac{1}{3}, b^2 = \frac{1}{12}$

$\therefore$ Equation of ellipse is $\frac{x^2}{\frac{1}{9}} + \frac{y^2}{\frac{1}{12}} = 1$

i.e., $9x^2 + 12y^2 = 1$

4. (d): Given ellipse is, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

and line is $y = x - 1$

Let the point of contact be $(2\cos\theta, \sqrt{3}\sin\theta)$.

$\therefore$ Equation of tangent becomes $\frac{x}{2}\cos\theta + \frac{y}{\sqrt{3}}\sin\theta = 1$

$$\Rightarrow \frac{\cos\theta}{2} = \frac{\sin\theta}{-\sqrt{3}} = 1$$

$\Rightarrow \cos\theta = 2, \sin\theta = -\sqrt{3}$

which is not possible. Thus, the given line can't be a tangent to the given ellipse.

5. (b): Given ellipse is, $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Let the foot of normal be $(3\cos\theta, 2\sin\theta)$

Equation of normal at this point is,

$$\frac{3x}{\cos\theta} - \frac{2y}{\sin\theta} = 5$$

Slope $= \frac{3}{2}\tan\theta = -2$

$$\Rightarrow \tan\theta = -\frac{4}{3}$$

or $\sin\theta = \frac{4}{5}, \cos\theta = -\frac{3}{5}$

Thus foot of normal be $\left(-\frac{9}{5}, \frac{8}{5}\right)$.

6. (c): Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ Equation of tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$\Rightarrow A \equiv (a\sec\theta, 0), B \equiv (0, b\,\mathrm{cosec}\theta)$

If the mid-point of segment AB be $Q(h, k)$, then

$$2h = a\sec\theta, 2k = b\,\mathrm{cosec}\theta$$

$$\Rightarrow \frac{a^2}{4h^2} + \frac{b^2}{4k^2} = 1$$

$\therefore$ Locus of Q is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$

7. (a): Let $P \equiv (a\cos\theta, b\sin\theta)$

$\therefore$ Equation of normal is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

$$\Rightarrow A \equiv \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right) \text{ and } B \equiv \left(0, \frac{a^2 - b^2}{b}\sin\theta\right)$$

If (h, k) be the mid-point of segment AB, then

$$2h = \frac{a^2 - b^2}{a}\cos\theta,\ 2k = \frac{a^2 - b^2}{b}\sin\theta$$

$$\Rightarrow 4h^2a^2 + 4k^2b^2 = (a^2 - b^2)^2$$

Thus, required locus is

$$4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2$$

8. (d): Let the point of contact be $P(a\cos\theta, b\sin\theta)$.

Then equation of tangent is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

Thus on comparing, $\frac{\cos\theta}{am} = \frac{\sin\theta}{-b} = -\frac{1}{c}$

$$\Rightarrow \cos\theta = -\frac{am}{c}, \sin\theta = \frac{b}{c}$$

Thus the point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

9. (a): Let the foot of normal be $P(a\cos\theta, b\sin\theta)$

Equation of normal at P is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

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$$\therefore \quad \frac{m\cos\theta}{a}=\frac{\sin\theta}{b}=\frac{c}{(b^2-a^2)}$$

$$\Rightarrow \quad \cos\theta=\frac{ac}{m(b^2-a^2)},\ \sin\theta=\frac{bc}{(b^2-a^2)}$$

Thus the foot of normal is $\left(\frac{a^2c}{m(b^2-a^2)}, \frac{b^2c}{(b^2-a^2)}\right)$

10. (d) : Let the mid-point of chord be $P(h, k)$. Equation of this chord will be $T = S_1$

i.e., $\frac{xh}{a^2}+\frac{yk}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$

$$\Rightarrow \quad \frac{h}{2a^2}=-\frac{k}{b^2}=-\frac{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)}{c} \Rightarrow \quad k=-\frac{b^2}{2a^2}\cdot h$$

Thus the required locus is $y=-\frac{b^2}{2a^2}x$ or $2a^2y + b^2x = 0$

11. (c) : Let $A \equiv (1, 0)$, $B \equiv (-1, \sqrt{12})$

$\therefore$ Given expression is, $PA + PB = a$

Here, $AB = \sqrt{4+12} = 4$

Thus $P(x, y)$ will lie on an ellipse having foci as A and B, provided $a > 4$

12. (d) : Clearly, $6 - a > 0$, $a - 2 > 0$

and $6 - a \neq a - 2$

$\Rightarrow \quad a \in (2, 6) - \{4\}$

13. (b) : Equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$

$$\Rightarrow \quad e^2=1-\frac{16}{25}=\frac{9}{25} \Rightarrow \quad e=\frac{3}{5}$$

Thus, foci are $\left(5\cdot\frac{3}{5}, 0\right)$ and $\left(-5\cdot\frac{3}{5}, 0\right)$

i.e. $(\pm 3, 0)$

14. (d) : If $P(x, y)$ be any point on the ellipse, then $PS = e \cdot PM$, where PM is perpendicular distance of P from directrix.

$$\Rightarrow \quad (x-1)^2+y^2=\frac{1}{2}\left(\frac{x+y-2}{\sqrt{2}}\right)^2$$

Thus required equation is $3x^2 + 3y^2 - 2xy - 4x + 4y = 0$

15. (a) : Given equation can be rewritten as

$$\frac{\left(\frac{x+y-5}{\sqrt{2}}\right)^2}{1}+\frac{\left(\frac{x-y+7}{\sqrt{2}}\right)^2}{3/2}=1$$

It is clearly an ellipse whose major and minor axes are along the lines

$x - y + 7 = 0$ and $x + y - 5 = 0$

Thus, centre of this ellipse is the intersection point of these lines *i.e.*, the point $(-1, 6)$

16. (b) : $OS_1 = ae = 6$, $OC = b$ (say), $CS_1 = a$

Area of $\Delta OCS_1 = \frac{1}{2}\cdot(OS_1)(OC) = 3b$

Semi perimeter of ΔOCS_1

$$= \frac{1}{2}(OS_1 + OC + CS_1) = \frac{1}{2}(6 + a + b)$$

Now, inradius of $\Delta OCS_1 = \frac{\text{Area of } \Delta OCS_1}{\text{Perimeter of } \Delta OCS_1}$

$$= \frac{3b}{\frac{1}{2}(6+a+b)} = 1 \text{ (given)}$$

$\Rightarrow \quad 5b = 6 + a$

Also $b^2 = a^2 (1 - e^2) = a^2 - 36$

Thus, $25b^2 = 36 + a^2 + 12a$

$\Rightarrow \quad 25(a^2 - 36) = 36 + a^2 + 12a$

$\Rightarrow \quad 2a^2 - a - 78 = 0 \Rightarrow \quad a=\frac{13}{2}, b=\frac{5}{2}$

Hence, area of ellipse $= \pi ab = \frac{65\pi}{4}$ sq. units.

17. (d) : In this case, base S_1S_2 is fixed and $PS_1 + PS_2$ is fixed. Hence area will be maximum when $PS_1 = PS_2$

$\therefore$ Maximum area $= \frac{1}{2}(2ae)b = abe$ sq. units.

18. (d) : Here, $S_1S_2 = 2ae$,

$PS_1 + PS_2 = 2a$

If the normal to ellipse at P, meets the x-axis at Q, then incentre of ΔPS_1S_2 will lie on segment PQ and will divide it in the ratio $\frac{PS_1+PS_2}{S_1S_2}$ *i.e.*, $\frac{1}{e}$

Let $P \equiv (a\cos\theta, b\sin\theta)$

Equation of normal at 'P' is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2=a^2e^2$$

$\Rightarrow \quad Q \equiv (ae^2\cos\theta, 0)$

Let $R(h, k)$ be the incenter, then

$$h=\frac{a\cos\theta\cdot e+ae^2\cos\theta}{1+\frac{1}{e}},\ k=\frac{b\sin\theta\cdot e+0\cdot1}{1+\frac{1}{e}}$$

$$\Rightarrow \quad h\left(1+\frac{1}{e}\right)=(ae+ae^2)\cos\theta \text{ and } be\cdot\sin\theta=k\left(1+\frac{1}{e}\right)$$

Thus locus of R is

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$$\frac{x^2\left(1+\frac{1}{e}\right)^2}{a^2e^2(1+e)^2}+\frac{y^2\left(1+\frac{1}{e}\right)^2}{b^2e^2}=1$$

which is clearly an ellipse.

19. (a) : $S_1 = (ae, 0)$, $S_2 = (-ae, 0)$

Let the tangent be

$$xb\cos\theta + ay\sin\theta - ab = 0$$

$$S_1P_1 = \frac{|abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$\Rightarrow \quad S_1P_1 = \frac{ab(1-e\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

Similarly, $S_2P_2 = \dfrac{ab(1+e\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$$\Rightarrow \quad (S_1P_1)\cdot(S_2P_2) = \frac{a^2b^2(1-e^2\cos^2\theta)}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{a^2b^2(1-e^2\cos^2\theta)}{a^2+(b^2-a^2)\cos^2\theta} = \frac{a^2b^2(1-e^2\cos^2\theta)}{a^2-a^2e^2\cos^2\theta} = b^2$$

which is clearly a constant

Thus, $(S_1P_1)\,(S_2P_2) = b^2$

20. (d) : Let the equation of ellipse having centre $(1, 2)$ be

$$\frac{(x-1)^2}{a^2}+\frac{(y-2)^2}{b^2}=1 \qquad ...(i)$$

We have, distance between focus and centre,

$ae = (1 - 6)^2 + (2 - 2)^2 = 5 \Rightarrow a^2e^2 = 25$

Also $(4, 6)$ lies on (i)

$$\therefore \quad \frac{9}{a^2}+\frac{16}{b^2}=1$$

Now, $b^2 = a^2(1 - e^2) = a^2 - 25$

$$\Rightarrow \frac{9}{a^2}+\frac{16}{a^2-25}=1 \Rightarrow a^4 - 50a^2 + 225 = 0$$

$\Rightarrow \quad a^2 = 5,\ 45$

But $a^2 = 5$ is clearly rejected

$\therefore \quad a^2 = 45 \Rightarrow b^2 = 20$

Thus, required equation is $\dfrac{(x-1)^2}{45}+\dfrac{(y-2)^2}{20}=1$

21. (d) : Tangent to the ellipse is

$$y = mx \pm \sqrt{\frac{5m^2}{3}+\frac{5}{2}}$$

If it passes through $P(1, 2)$, then $(2 - m)^2 = \dfrac{5m^2}{3}+\dfrac{5}{2}$

$\Rightarrow \quad 4m^2 + 24m - 9 = 0$

If it's roots are m_1 and m_2, then

$$m_1 + m_2 = -6,\ m_1m_2 = -\frac{9}{4}$$

$$\Rightarrow \quad |m_1 - m_2| = \sqrt{36+9} = 3\sqrt{5}$$

$$\therefore \quad \tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| = \frac{3\sqrt{5}}{\left|1-\frac{9}{4}\right|} = \frac{12}{\sqrt{5}}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{5}}\right)$$

22. (c) : Equation of normal at point 'θ' is

$$\frac{5x}{\cos\theta}-\frac{3y}{\sin\theta}=16$$

On comparing with $5x - 3y = 8\sqrt{2}$, we get

$$\Rightarrow \quad \cos\theta = \sin\theta = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

23. (c) : Let the point 'P' be $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

A.T.Q.

$$6\cos^2\theta + 2\sin^2\theta = 4$$

$$\Rightarrow \quad 6 - 4\sin^2\theta = 4 \Rightarrow \sin^2\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{4} \quad (\because\ \theta \text{ lies in first quadrant})$$

24. (b) : Equation of tangent at point 'θ' is,

$$\frac{x}{4}\cos\theta + \frac{\sqrt{11}}{16}y\sin\theta = 1$$

It will touch the given circle, if

$$\frac{\left|\frac{\cos\theta}{4}-1\right|}{\sqrt{\frac{\cos^2\theta}{16}+\frac{11\sin^2\theta}{256}}} = 4$$

$$\Rightarrow \quad 4\cos^2\theta + 8\cos\theta - 5 = 0$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

25. (c) : The given distance is clearly the length of semi major axis.

Thus, $\sqrt{\dfrac{a^2+2b^2}{2}} = a$

$$\Rightarrow \quad 2b^2 = a^2 \Rightarrow 2a^2(1 - e^2) = a^2$$

$$\Rightarrow \quad e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$$

26. (a) : Let line AB be $bx\cos\theta + ay\sin\theta - ab = 0$

Now, combined equation of lines OA and OB is

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$$x^2 + y^2 = a^2\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right)^2$$

i.e., $x^2 \cdot \sin^2\theta + y^2\left(1 - \frac{a^2}{b^2}\sin^2\theta\right) - \frac{a}{b}\sin 2\theta \cdot xy = 0$

Since the lines OA and OB are mutually perpendicular, hence

$$\sin^2\theta + 1 - \frac{a^2}{b^2}\sin^2\theta = 0$$

$$\Rightarrow \sin^2\theta\left(\frac{a^2}{b^2} - 1\right) = 1 \Rightarrow \sin^2\theta\left(\frac{1}{1-e^2} - 1\right) = 1$$

$$\Rightarrow e^2 \cdot \sin^2\theta = 1 - e^2 \Rightarrow e^2(1 + \sin^2\theta) = 1$$

27. (a) : Let the foot of normal is $(a\cos\theta, b\sin\theta)$, then $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

and the given lines will be identical,

Thus, $\frac{m\cos\theta}{a} = \frac{\sin\theta}{b} = \frac{c}{(b^2 - a^2)}$

$$\Rightarrow \cos\theta = \frac{ac}{m(b^2 - a^2)},\ \sin\theta = \frac{bc}{(b^2 - a^2)}$$

$$\Rightarrow \frac{a^2}{m^2} + b^2 = \frac{(b^2 - a^2)^2}{c^2}$$

28. (c) : If the point of intersection be $R(h, k)$, then

$$h = \frac{a\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)},\ k = \frac{b\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$$

$$\Rightarrow h = \frac{a\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\frac{1}{2}},\ k = \frac{b\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\frac{1}{2}}$$

Thus, $\frac{h^2}{a^2} + \frac{k^2}{b^2} = 4$

Hence, locus is, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$

29. (d) : Let $P \equiv (a\cos\theta, b\sin\theta)$

$\Rightarrow P_1 \equiv (a\cos\theta, 0),\ P_2 \equiv (0, b\sin\theta)$

Thus, equation of line P_1P_2 is $\frac{x}{a\cos\theta} + \frac{y}{b\sin\theta} = 1$

$$\Rightarrow \frac{x/a}{\cos(-\theta)} - \frac{y/b}{\sin(-\theta)} = 1$$

which is clearly a normal to the ellipse of the form $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, where $\frac{A}{A^2 - B^2} = \frac{\lambda}{a}$ and $\frac{B}{A^2 - B^2} = \frac{\lambda}{b}$

If $a > b$, then $B > A$

Let the eccentricity of the second ellipse be e_1

$$\Rightarrow 1 - e_1^2 = \frac{A^2}{B^2} = \frac{b^2}{a^2} = 1 - e^2 \Rightarrow e_1 = e$$

30. (a) : Equation of normal at $\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{a(x - ae)}{e} = (ay - b^2)$$

It should pass through $(0, -b)$

$$\Rightarrow \frac{a(0 - ae)}{e} = -ab - b^2$$

$$\Rightarrow a^2 = ab + b^2 \Rightarrow \frac{b^2}{a^2} + \frac{b}{a} - 1 = 0$$

$$\Rightarrow \frac{b}{a} = \frac{-1 + \sqrt{5}}{2} \Rightarrow \frac{b^2}{a^2} = \frac{3 - \sqrt{5}}{2} = 1 - e^2$$

$$\Rightarrow e^2 = \frac{\sqrt{5} - 1}{2} \Rightarrow e = \sqrt{\frac{\sqrt{5} - 1}{2}}$$

31. (c) : Tangent for $y^2 = 4x$ is $y = mx + \frac{1}{m}$

and for $\frac{x^2}{8} + \frac{y^2}{2} = 1$ is $y = mx \pm \sqrt{8m^2 + 2}$

For common tangents we must have

$$\frac{1}{m} = \pm\sqrt{8m^2 + 2} \Rightarrow m = \pm\frac{1}{2}$$

Thus, common tangents are

$$y = \frac{x}{2} + 2 \text{ and } y = -\frac{x}{2} - 2$$

i.e. $2y - x - 4 = 0$ and $2y + x + 4 = 0$

32. (a) : Clearly, the tangents have been drawn by taking any point on the director circle of the ellipse. Any point on the director circle can be taken as $(\sqrt{13}\cos\theta, \sqrt{13}\sin\theta)$.

Equation of corresponding chord of contact is

$$\frac{\sqrt{13}}{9}.\cos\theta.x + \frac{\sqrt{13}}{4}\sin\theta\, y = 1$$

$$\therefore \quad \text{Distance from the origin} = \frac{1}{\sqrt{\frac{13}{81}\cos^2\theta + \frac{13}{16}\sin^2\theta}}$$

$$= \frac{36}{\sqrt{1053 - 845\cos^2\theta}} = \frac{36}{\sqrt{208}} = \frac{9}{\sqrt{13}}$$

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33. (b) : Equation of AB is $T = 0$,

i.e. $\frac{5y}{9}=1 \Rightarrow y=\frac{9}{5}$

Putting $y=\frac{9}{5}$ in the equation of ellipse, we get

$$\frac{x^2}{16}+\frac{81}{25\cdot 9}=1 \Rightarrow x=\pm\frac{16}{5}. \text{ Thus, } AB=\frac{32}{5}$$

Distance of P from $AB = 5 - \frac{9}{5}=\frac{16}{5}$

Thus, area of $\Delta PAB = \frac{1}{2}\cdot\frac{32}{5}\cdot\frac{16}{5}=\frac{256}{25}$ sq. units

34. (a) : Other common tangent will clearly be the reflection of $x - 2y + 4 = 0$ in the x-axis.
Thus other common tangent will be $x + 2y + 4 = 0$

35. (c) : Let the eccentric angle of point 'P' be θ, then equation of tangent and normal at point are respectively

$$\frac{x}{6}\cos\theta+\frac{y}{3}\sin\theta=1 \text{ and } \frac{6x}{\cos\theta}-\frac{3y}{\sin\theta}=27$$

Thus, $A \equiv \left(0, \frac{3}{\sin\theta}\right)$ and $B \equiv \left(\frac{27}{6}\cos\theta, 0\right)$

Area of $\Delta OAB = \frac{1}{2}\cdot\frac{3}{\sin\theta}\cdot\frac{27}{6}\cos\theta=\frac{27}{4}$ (given)

$\Rightarrow \cot\theta = 1 \Rightarrow \theta=\frac{\pi}{4}, \frac{5\pi}{4}$

36. (c) : Let $P \equiv (2\cos\theta, \sin\theta)$

Equation of tangent at 'P' is $\frac{x}{2}\cos\theta + y\sin\theta = 1$
$\Rightarrow A \equiv (2\sec\theta, 0)$

Equation of normal at 'P' is $\frac{2x}{\cos\theta}-\frac{y}{\sin\theta}=3$
$\Rightarrow B\equiv\left(\frac{3}{2}\cos\theta, 0\right)$

Now, $AB=\sqrt{\left(2\sec\theta-\frac{3}{2}\cos\theta\right)^2}=2$ (given)

$\Rightarrow 4 - 3\cos^2\theta = 4\cos\theta \Rightarrow 3\cos^2\theta + 4\cos\theta - 4 = 0$

$\Rightarrow \cos\theta=\frac{2}{3}$

37. (a) : Equation of chord is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$\Rightarrow A\equiv\left(\frac{a\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)},0\right)$$

Since $OA = c$, therefore,

$$a\cos\left(\frac{\theta_1-\theta_2}{2}\right)=c.\cos\left(\frac{\theta_1+\theta_2}{2}\right)$$

$$\Rightarrow \cos\frac{\theta_1}{2}.\cos\frac{\theta_2}{2}(a-c)=\sin\frac{\theta_1}{2}.\sin\frac{\theta_2}{2}(-c-a)$$

$$\Rightarrow \tan\frac{\theta_1}{2}.\tan\frac{\theta_2}{2}=\frac{c-a}{c+a}$$

38. (b) : Equation of tangent at point

$(\sqrt{18}\cos\theta, \sqrt{32}\sin\theta)$, is $\frac{x}{\sqrt{18}}\cos\theta+\frac{y}{\sqrt{32}}\sin\theta=1$

Its slope $=-\frac{\sqrt{32}}{\sqrt{18}}.\cot\theta=-\frac{4}{3}$ (given)

$\Rightarrow \cot\theta = 1$

Also, $A\equiv(\sqrt{18}\sec\theta, 0), B\equiv(0, \sqrt{32}\,\text{cosec}\,\theta)$

$$\Rightarrow \Delta_{OAB}=\frac{1}{2}\left|3\sqrt{2}\sec\theta\cdot 4\sqrt{2}\text{cosec}\,\theta\right|$$

$= 12\cdot\sqrt{2}\cdot\sqrt{2} = 24$ sq. units

39. (a) : Any tangent to $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ is,

$$y=mx\pm\sqrt{(a^2+b^2)m^2+b^2}$$

It will also be a tangent to second ellipse if

$$a^2m^2+(a^2+b^2)=(a^2+b^2)\,m^2+b^2$$

$$\Rightarrow b^2m^2=a^2 \Rightarrow m=\pm\frac{a}{b}$$

Thus common tangents are

$$y=\frac{a}{b}x\pm\sqrt{(a^2+b^2).\frac{a^2}{b^2}+b^2}$$

and $y=-\frac{a}{b}x\pm\sqrt{(a^2+b^2)\frac{a^2}{b^2}+b^2}$

i.e., $by=ax\pm\sqrt{a^4+b^4+a^2b^2}$

and $by=-ax\pm\sqrt{a^4+b^4+a^2b^2}$

40. (a) : Let the extremities of the chord be $P_1 \equiv (a\cos\theta, b\sin\theta)$ and $P_2 \equiv (-a\sin\theta, b\cos\theta)$

Now, $(P_1P_2^2) = a^2(\cos\theta+\sin\theta)^2 + b^2(\sin\theta-\cos\theta)^2$
$= a^2 + b^2 + (a^2 - b^2)2\sin\theta\cos\theta$
$\le a^2 + b^2 + a^2 - b^2 = 2a^2$

$\Rightarrow P_1P_2 \le a\sqrt{2}$

41. (d) : Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

If the line $y = mx + c$ be its tangent, then
$c^2 = a^2m^2 + b^2$

Tangent is given to be $x + 4y - 10 = 0$

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$\Rightarrow \frac{m}{1}=-\frac{1}{4}=\frac{c}{-10} \Rightarrow m=-\frac{1}{4}, c=\frac{5}{2} \therefore \frac{25}{4}=\frac{a^2}{16}+b^2$

$\Rightarrow a^2+16b^2=100$

We also have $P(4,-1)$ to be a point on the ellipse,

$\therefore \frac{16}{a^2}+\frac{1}{b^2}=1$

$\Rightarrow a^2+16b^2=a^2b^2$

$\Rightarrow a^2b^2=100 \Rightarrow ab=10$

$\therefore$ Area of ellipse $=\pi ab=10\pi$ sq. units

42. (b) : If 'O' be the center of ellipse then for triangle S_1S_2B to be right angled, we must have

$\angle OBS_1=\angle OBS_2=\frac{\pi}{4}$.

Thus, $OS_1=OB=OS_2 \Rightarrow ae=b \Rightarrow a^2e^2=a^2-a^2e^2$

$\Rightarrow e^2=\frac{1}{2} \Rightarrow e=\frac{1}{\sqrt{2}}$

43. (c) : Let $P\equiv(h,k)$

Equation of chord of contact is $\frac{xh}{a^2}+\frac{yk}{b^2}=1$

i.e., $xb^2h+ya^2k-a^2b^2=0$

It will touch the circle $x^2+y^2=c^2$, if

$$\frac{a^2b^2}{\sqrt{b^4h^2+a^4k^2}}=c$$

Hence, locus of point 'P' is

$$b^4x^2+a^4y^2=\frac{a^4b^4}{c^2} \quad i.e., \quad \frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}$$

44. (c) : Slope of tangent must be $-\frac{9}{8}$.

Slope of tangent at any point $P(\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $-\frac{b}{a}\cot\theta$

Thus, $-\frac{9}{8}=-\frac{2}{3}\cot\theta \Rightarrow \tan\theta=\frac{16}{27}$

$\Rightarrow \cos\theta=\frac{27}{\sqrt{985}}, \sin\theta=\frac{16}{\sqrt{985}}$

or $\cos\theta=-\frac{27}{\sqrt{985}}, \sin\theta=-\frac{16}{\sqrt{985}}$

Thus, required point is

$$\left(\frac{27}{2\sqrt{985}},\frac{16}{3\sqrt{985}}\right) \text{ or } \left(-\frac{27}{2\sqrt{985}},-\frac{16}{3\sqrt{985}}\right)$$

45. (b) : Let $P\equiv(h,k)$

Any tangent to the ellipse is

$y=mx\pm\sqrt{16m^2+9}$

If it passes through 'P', then

$(k-mh)^2=16m^2+9$

$\Rightarrow m^2(h^2-16)-2mhk+(k^2-9)=0$

If its roots are m_1 and m_2, then

$$m_1m_2=\frac{k^2-9}{h^2-16}$$

Since, the drawn tangents meet the coordinate axes at concyclic points, thus,

$m_1m_2=1 \Rightarrow k^2-9=h^2-16$

Hence required locus is,

$x^2-y^2=7$.

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CHALLENGING PROBLEMS

For Entrance Exams

1. The value of $\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ is

(a) 1 (b) $\frac{\sqrt{7}}{2}$ (c) $\frac{3\sqrt{3}}{4}$ (d) $\frac{\sqrt{15}}{4}$

2. The equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ can be solved if

(a) $-\sqrt{3} \le a \le \sqrt{3}$ (b) $-\sqrt{2} \le a \le \sqrt{2}$
(c) $-1 \le a \le 1$ (d) None of these

3. Given ΔABC is inscribed in the semicircle with diameter AB. The area of ΔABC equals 2/9 of the area of the semicircle. If the measure of the smallest angle in ΔABC is x, then $\sin 2x$ is equal to

(a) $\frac{\pi}{9}$ (b) $\frac{2\pi}{9}$ (c) $\frac{\pi}{18}$ (d) $\frac{\pi}{8}$

4. Set of all the values of x satisfying the inequality $\log_{x+\frac{1}{x}}\left(\log_2\frac{x-1}{x+2}\right) > 0$ is

(a) $(-5, -2)$ (b) $(2, 5)$
(c) $(5, \infty)$ (d) ϕ

5. The integer value of x for which $x^2 + 19x + 92$ is square of an integer are

(a) -8 and -11 (b) -8 and 11
(c) 8 and 11 (d) None of these

6. Let a_n be the n^{th} term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \ne \beta$, then the common ratio is

(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$

7. The coefficient of $x^n y^n$ in the expansion of $\{(1 + x)(1 + y)(x + y)\}^n$ is

(a) $\sum_{r=0}^{n} C_r^2$ (b) $\sum_{r=0}^{n} C_r^3$

(c) $\sum_{r+s=0}^{n} {}^nC_r\,{}^nC_s^2$ (d) None of these

8. Given 6 different toys of red colour, 5 different toys of blue colour and 4 different toys of green colour. Combination of toys that can be chosen taking at least one red and one blue toys are

(a) 31258 (b) 31248
(c) 31268 (d) None of these

9. If length of common chord of two circles $x^2 + y^2 + 8x + 1 = 0$ and $x^2 + y^2 + 2\mu y - 1 = 0$ is $2\sqrt{6}$, then value of μ are

(a) ± 2 (b) ± 3
(c) ± 4 (d) None of these

10. If two distinct chords drawn from the point (4, 4) on the parabola $y^2 = 4x$ are bisected on the line $y = mx$, then the set of value of m is given by

(a) $\left(\frac{1-\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2}\right)$ (b) R

(c) $(0, \infty)$ (d) $(-2, 2)$

11. If the focal distance of an end of the minor axis of any ellipse (its axes as x and y axis respectively) is k and the distance between the foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2}+\frac{y^2}{h^2}=1$ (b) $\frac{x^2}{k^2}+\frac{y^2}{k^2-h^2}=1$

(c) $\frac{x^2}{k^2}-\frac{y^2}{k^2-h^2}=1$ (d) $\frac{x^2}{k^2}+\frac{y^2}{k^2+h^2}=1$

12. If $A = \left\{x : \frac{\pi}{6} \le x \le \frac{\pi}{3}\right\}$ and $f(x) = \cos x - x(1 + x)$, then $f(A)$ is equal to

(a) $\left[\frac{1}{2}-\frac{\pi}{3}-\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}-\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(b) $\left[\frac{1}{2}+\frac{\pi}{3}-\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}+\frac{\pi}{6}-\frac{\pi^2}{36}\right]$

(c) $\left(\frac{1}{2}-\frac{\pi}{3}+\frac{\pi^2}{9}, \frac{\sqrt{3}}{2}-\frac{\pi}{6}+\frac{\pi^2}{36}\right)$

(d) None of these

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13. If the function $f(x)=\left[\frac{(x-2)^3}{a}\right]\sin(x-2)+a\cos(x-2)$, (where $[\cdot]$ denotes the greatest integer function) is continuous and differentiable in (4, 6), then

(a) $a \in [8, 64]$ (b) $a \in (0, 8]$

(c) $a \in [64, \infty)$ (d) None of these

14. If $f(x)=\begin{cases}1/|x|; & |x|\geq 1\\ ax^2+b; & -1<x<1\end{cases}$ is differentiable $\forall\, x$, then value of a and b is

(a) $a = 1/2$, $b = -3/2$ (b) $a = -1/2$, $b = 3/2$

(c) $a = 3/2$, $b = 1/2$ (d) None of these

15. If $f''(x) > 0$ and $f'(1) = 0$ such that $g(x) = f(\cot^2 x + 2\cot x + 2)$, where $0 < x < \pi$, then the interval in which $g(x)$ is decreasing is

(a) $(0, \pi)$ (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(\frac{3\pi}{4}, \pi\right)$ (d) $\left(0, \frac{3\pi}{4}\right)$

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16. If $I=\int\frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx$, then I equals

(a) $\log|x|+\log\left|1+\sqrt{1-x^2}\right|+\sin^{-1}\sqrt{x}+C$

(b) $\log|x|-\log\left|1-\sqrt{1-x^2}\right|+\tan^{-1}x+C$

(c) $\log|x|-\log\left|1+\sqrt{1-x^2}\right|-\sin^{-1}x+C$

(d) $\log\left|1+\sqrt{1-x^2}\right|-\log|x|+\cos^{-1}x+C$

17. If $f(x)=\tan^{-1}x+\ln\sqrt{1+x}-\ln\sqrt{1-x}$. The integral of $(1/2)f'(x)$ with respect to x^4 is

(a) $e^{-x^4}+c$ (b) $-\ln(1-x^4)+c$

(c) $e^{\sqrt{1-x^4}}+c$ (d) $\ln(1+x^2)+c$

18. The value of $\int_1^{16}\tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ is

(a) $\frac{16\pi}{3}+2\sqrt{3}$ (b) $\frac{4}{3}\pi-2\sqrt{3}$

(c) $\frac{4}{3}\pi+2\sqrt{3}$ (d) $\frac{16}{3}\pi-2\sqrt{3}$

19. If f is a continuous function such that $\int_0^x f(t)dt\to\infty$ as $|x|\to\infty$, then for all $k\in R$, then $k^2x^2+\int_0^x f(t)dt-a=0 (a>0)$ has

(a) all roots in $(-\infty, 0)$

(b) all roots in $(0, \infty)$

(c) odd number of roots in $(-\infty, 0)$ and odd number of roots in $(0, \infty)$

(d) None of these

20. Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the coordinate axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of

(a) circles (b) parabolas

(c) hyperbolas (d) straight lines

21. The area bounded by the curve $y = x^4 - 2x^3 + x^2 + 3$, the axis of abscissa and two ordinates corresponding to the points of minimum of the function $y(x)$ (in sq. units) is

(a) 10/3 (b) 27/10

(c) 21/10 (d) None of these

22. Solution of the equation $y=x\frac{dy}{dx}+\frac{dx}{dy}$ represents

(a) Family of straight lines and a parabola

(b) Family of straight lines and a hyperbola

(c) Family of circles and parabola

(d) None of these

23. The differential equation of family of parabola with foci at the origin and axis along the x axes is

(a) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0$

(b) $x\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}-y=0$

(c) $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}+y=0$

(d) None of these

24. If z_1, z_2 are two complex numbers such that $\left|\frac{z_1-z_2}{z_1+z_2}\right|=1$ and $iz_1 = Kz_2$, where $K\in R$, then the angle between $z_1 - z_2$ and $z_1 + z_2$ is

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(a) $\tan^{-1}\left(\frac{2K}{K^2+1}\right)$ (b) $\tan^{-1}\left(\frac{K}{1-K^2}\right)$

(c) $-2\tan^{-1}K$ (d) $2\tan^{-1}K$

25. If α is a complex n^{th} root of unity and if Z_1 and Z_2 are two complex numbers, then $\sum_{r=0}^{n-1}|Z_1+\alpha^r Z_2|^2 =$

(a) $n^2|Z_1+Z_2|^2$ (b) $\left(\frac{Z_1}{n}+\frac{Z_2}{n}\right)^2$

(c) $n(|Z_1|^2+|Z_2|^2)$ (d) $n^2(|Z_1|^2+|Z_2|^2)$

26. Two persons A and B throw a die alternately till one of them get a "six" and wins the game. The probability of winning of B if A starts first is

(a) $\frac{6}{11}$ (b) $\frac{5}{11}$

(c) $\frac{3}{11}$ (d) None of these

27. The values of λ for which the system of equations $x+y-3=0$, $(1+\lambda)x+(2+\lambda)y-8=0$, $x-(1+\lambda)y+(2+\lambda)=0$ is consistent are

(a) $-5/3, 1$ (b) $2/3, -3$

(c) $-1/3, -3$ (d) $0, 0$

28. The number of values of k for which the system of equations $(k+1)x+8y=4k$, $kx+(k+3)y=3k-1$ has no solution is

(a) 0 (b) 1 (c) 2 (d) infinite

29. If $A=\begin{bmatrix}0 & 5\\ 0 & 0\end{bmatrix}$ and $f(x)=1+x+x^2+\ldots+x^{16}$, then $f(A)$ is equal to

(a) 0 (b) $\begin{bmatrix}1 & 5\\ 0 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 5\\ 0 & 0\end{bmatrix}$ (d) $\begin{bmatrix}0 & 5\\ 1 & 1\end{bmatrix}$

30. If $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$ and $\vec{a}\perp(\vec{b}+\vec{c}), \vec{b}\perp(\vec{c}+\vec{a}), \vec{c}\perp(\vec{a}+\vec{b})$ then $|\vec{a}+\vec{b}+\vec{c}|$ is equal to

(a) 50 (b) 25

(c) $5\sqrt{2}$ (d) 12

31. If lines $\frac{x-1}{2}=\frac{y-2}{x_1}=\frac{z-3}{x_2}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ lies in the same plane then for equation $x_1t^2+(x_2+2)t+a=0$, which of the options holds?

(a) $2x_1-x_2=1$

(b) Sum of roots of given equation $=-2$

(c) $2x_1+x_2=-4$

(d) Sum of roots of given equation $=0$

32. The variance of the first n natural numbers is

(a) $\frac{n^2-1}{12}$ (b) $\frac{n^2-1}{6}$

(c) $\frac{n^2+1}{6}$ (d) $\frac{n^2+1}{12}$

SOLUTIONS

1. (b): Let $S=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ and $C=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}$

Now, $C+iS=\alpha+\alpha^2+\alpha^4$ ...(i)

where, $\alpha=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$, is complex 7^{th} root of unity

Also, $C-iS=\alpha^6+\alpha^5+\alpha^3$ ...(ii)

$\because \alpha^6=\bar{\alpha}, \alpha^5=\bar{\alpha}^2, \alpha^3=\bar{\alpha}^4$

Adding eqn. (i) and (ii), we get

$2C=\alpha+\alpha^2+\alpha^3+\ldots+\alpha^6=\frac{\alpha^7-\alpha}{\alpha-1}=-1 \quad (\because \alpha^7=1)$

$\Rightarrow C=-1/2$

Multiplying eqn. (i) and (ii), we get

$C^2+S^2=2 \Rightarrow S=\frac{\sqrt{7}}{2}$

2. (b): We have, $\sin^4x-2\cos^2x+a^2=0$

or, $y^2-2(1-y)+a^2=0$ [where $\sin^2x=y$]

$\Rightarrow y^2+2y+a^2-2=0$

For y to be real, discriminant ≥ 0

$\Rightarrow 4-4(a^2-2)\geq 0$

$\Rightarrow a^2\leq 3$ ...(i)

Since, $\sin^2x=y \Rightarrow 0\leq y\leq 1$

$\Rightarrow 0\leq\sqrt{3-a^2}-1\leq 1$

$\Rightarrow 1\leq\sqrt{3-a^2}\leq 2 \Rightarrow 1\leq 3-a^2\leq 4$

$\Rightarrow 2-a^2\geq 0 \Rightarrow a^2\leq 2$

$\Rightarrow -\sqrt{2}\leq a\leq\sqrt{2}$.

3. (a): Since, $a^2+b^2=c^2=4r^2$ ...(i)

Also, $\frac{1}{2}a\cdot b=\frac{2}{9}\cdot\left(\frac{1}{2}\pi r^2\right)$

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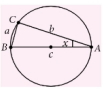
$\Rightarrow \ 9ab = 2\pi r^2$ ...(ii)

From eqs. (i) and (ii), we get



$$\frac{a^2+b^2}{9ab} = \frac{2}{\pi}$$

$$\Rightarrow \frac{a}{b}+\frac{b}{a} = \frac{18}{\pi}$$

Now, $\angle BAC = x$, then $\dfrac{\sin x}{\cos x}+\dfrac{\cos x}{\sin x} = \dfrac{18}{\pi}$

$$\Rightarrow \frac{\cos^2 x+\sin^2 x}{\sin x\cdot\cos x} = \frac{18}{\pi}$$

$$\Rightarrow \sin x\cdot\cos x = \frac{\pi}{18} \Rightarrow \sin 2x = \frac{\pi}{9}$$

4. (d): Since $\log_{x+\frac{1}{x}} \log_2\left[\dfrac{x-1}{x+2}\right] > 0$

$$\therefore \frac{x-1}{x+2} > 0$$

$\Rightarrow x > 1, x < -2$ ...(i)

But $x+\dfrac{1}{x} > 0$

$$\Rightarrow \frac{x^2+1}{x} > 0 \Rightarrow x > 0 \quad \text{...(ii)}$$

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From (i) and (ii), we get

$$x > 1; \log_{x+\frac{1}{x}}\left(\log_2\frac{x-1}{x+2}\right) > 0$$

Take antilog both sides,

$$\log_2\frac{x-1}{x+2} > 1$$

Take again antilog both sides,

$$\frac{x-1}{x+2} > 2 \Rightarrow \frac{x+5}{x+2} < 0$$

$\Rightarrow x \in (-5, -2)$ [But not true as $x > 1$]

5. (a): Let $x^2 + 19x + 92 = m^2$, $m \in I$

$x^2 + 19x + 92 - m^2 = 0$

$$\therefore \ x = \frac{-19 \pm \sqrt{4m^2-7}}{2}$$

Since, $x \in I$, so $-19 \pm \sqrt{4m^2-7}$ must be an even integer

$\therefore \ \sqrt{4m^2-7}$ must be an odd integer

Let $\sqrt{4m^2-7} = 2k+1; k \in I$

$\therefore \ 4m^2 - (2k+1)^2 = 7$

$(2m-2k-1)(2m+2k+1) = 7$

Since, 7 is prime, so possible cases are as follows :

(i) $2m - 2k - 1 = 1$ and $2m + 2k + 1 = 7$, then $m = 2$

(ii) $2m - 2k - 1 = 7$ and $2m + 2k + 1 = 1$, then $m = 2$

$\therefore$ At $m = 2$, $x = -8, -11$

(iii) $2m - 2k - 1 = -1$ and $2m + 2k + 1 = -7$, then $m = -2$

(iv) $2m - 2k - 1 = -7$ and $2m + 2k + 1 = -1$, then $m = -2$

$\therefore$ At $m = -2$, $x = -8, -11$

6. (a): Let the common ratio of a G.P. be r.

$$\text{Now, } \sum_{n=1}^{100} a_{2n} = a_2 + a_4 + a_6 + ... + a_{200}$$

$$= ar + ar^3 + ar^5 + ... + ar^{199} = \alpha \quad \text{...(i)}$$

$$\text{Also, } \sum_{n=1}^{100} a_{2n-1} = a_1 + a_3 + a_5 + ... + a_{199}$$

$$= a + ar^2 + ar^4 + ... + ar^{198} = \beta \quad \text{...(ii)}$$

On dividing eqn. (ii) by eqn. (i), we get

$$\frac{r(a+ar^2+ar^4+...+ar^{198})}{(a+ar^2+ar^4+...ar^{198})} = \frac{\alpha}{\beta}$$

$$\Rightarrow \ r = \frac{\alpha}{\beta}$$

7. (b): We have, $\{(1+x)(1+y)(x+y)\}^n$

$= (C_0 + C_1x + C_2x^2 + ... + C_nx^n) \times (C_0y^n + C_1y^{n-1} + ... + C_{n-1}y + C_n) \times (C_0y^n + C_1x^{n-1}y + ... + C_{n-1}xy^{n-1} + C_ny^n)$

$\therefore$ Coefficient of $x^n y^n = C_0^3 + C_1^3 + C_2^3 + ... + C_n^3 = \sum_{r=0}^{n} C_r^3$

8. (b): At least one red toy can be chosen in

$= {}^6C_1 + {}^6C_2 + ... + {}^6C_6 = 2^6 - 1 = 63$ ways

At least one blue toy can be chosen in

$= {}^5C_1 + {}^5C_2 + ... + {}^5C_5 = 2^5 - 1 = 31$ ways

Green toys (with no restriction) can be selected in

$= {}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 = 16$ ways

$\therefore$ Total ways of selection $= 63 \times 31 \times 16 = 31248$

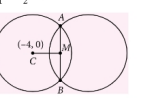
9. (b): Common chord is $S_1 - S_2 = 0$

$\Rightarrow \ 4x - \mu y + 1 = 0$

$$\therefore \ AC = \sqrt{15}$$

$$AM = \frac{AB}{2} = \sqrt{6}$$

$\therefore \ CM = 3$



$$\therefore \text{ Perpendicular distance} = \frac{|-16+1|}{\sqrt{16+\mu^2}} = 3$$

$\Rightarrow \ \mu = \pm 3$

10. (a): Any point on the line $y = mx$ can be taken as (t, mt).

Equation of the chord of parabola with (t, mt) as mid point

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$y\,mt - 2(x + t) = m^2t^2 - 4t$

Since, it passes through (4, 4).

$\therefore \quad 4mt - 2(4 + t) = m^2t^2 - 4t$

$\Rightarrow \quad m^2t^2 - 2(2m + 1)t + 8 = 0$

For two such chords $D > 0$

$\Rightarrow \quad (2m + 1)^2 - 8m^2 > 0$

$\Rightarrow \quad 4m^2 - 4m - 1 < 0 \Rightarrow \dfrac{1-\sqrt{2}}{2} < m < \dfrac{1+\sqrt{2}}{2}$

11. (b): Let equation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and e is eccentricity of ellipse.

$\therefore \quad 2h = 2ae \Rightarrow ae = h \qquad ...(i)$

Focal distance of one end of minor axis say $(0, b)$ is k,

$\therefore \quad a + e\,(0) = k \Rightarrow a = k \qquad ...(ii)$

From (i) and (ii), $b^2 = a^2\,(1 - e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2 = k^2 - h^2$

$\therefore \quad$ Equation of ellipse is $\dfrac{x^2}{k^2} + \dfrac{y^2}{k^2 - h^2} = 1.$

12. (a): We have $f(x) = \cos x - x\,(1 + x)$

Since, in the interval $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$, $\cos x$ decreases and $x(1 + x)$ increases.

$\therefore \quad f(x)$ decreases in $\left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$.

$\therefore \quad f\left(\dfrac{\pi}{3}\right) \le f(x) \le f\left(\dfrac{\pi}{6}\right); x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

Hence, $f(A) = \left[\dfrac{1}{2} - \dfrac{\pi}{3}\left(1 + \dfrac{\pi}{3}\right), \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}\left(1 + \dfrac{\pi}{6}\right)\right]$

$= \left[\dfrac{1}{2} - \dfrac{\pi}{3} - \dfrac{\pi^2}{9}, \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6} - \dfrac{\pi^2}{36}\right]$

13. (c): We have, $x \in (4, 6)$

$\Rightarrow \quad 2 < x - 2 < 4$

$\Rightarrow \quad \dfrac{8}{a} < \dfrac{(x-2)^3}{a} < \dfrac{64}{a}, \quad a > 0$

For $f(x)$ to be continuous and differentiable in (4, 6), $\left[\dfrac{(x-2)^3}{a}\right]$ must attain a constant value for $x \in (4, 6)$.

Clearly, this is possible only when $a \ge 64$

In that case, we have

$f(x) = a\cos(x - 2)$ which is continuous and differentiable.

$\therefore \quad a \in [64, \infty)$

14. (b): $f(x) = \begin{cases} 1/|x|; & |x| \ge 1 \\ ax^2 + b; & -1 < x < 1 \end{cases}$

$$f(x) = \begin{cases} -\dfrac{1}{x}; & x \le -1 \\ ax^2 + b; & -1 < x < 1 \\ \dfrac{1}{x}; & x \ge 1 \end{cases}$$

Since $f(x)$ is continuous at $x = 1$.

$\therefore \quad a \cdot 1 + b = 1 \qquad ...(i)$

Also, $f(x)$ is differentiable at $x = 1$.

$\therefore \quad 2ax = -\dfrac{1}{x^2}$

At $x = 1$, $2a = -1 \Rightarrow a = -1/2$

Using (i), we get, $b = 3/2$

15. (d): Here, $g(x) = f(\cot^2 x + 2\cot x + 2)$

$\Rightarrow \quad g'(x) = f'(\cot^2 x + 2\cot x + 2)$

$\{-2\cot x \operatorname{cosec}^2 x - 2\operatorname{cosec}^2 x\}$

For $g(x)$ to be decreasing, $g'(x) < 0$

$\Rightarrow \quad f'\{(\cot x + 1)^2 + 1\} \cdot (-2\operatorname{cosec}^2 x)\,(\cot x + 1) < 0$

$\Rightarrow \quad f'\{(\cot x + 1)^2 + 1\} \cdot (\cot x + 1) > 0 \qquad ...(i)$

$\{\text{as } f''(x) > 0 \Rightarrow f'(x) \text{ is increasing, then}$

$f'\{(\cot x + 1)^2 + 1\} > f'(1) = 0\ \forall x \in \left(0, \dfrac{3\pi}{4}\right) \cup \left(\dfrac{3\pi}{4}, \pi\right)$

Thus, equation (i) holds, if $\cot x + 1 > 0$

$\Rightarrow \quad \cot x > -1\ \forall x \in \left(0, \dfrac{3\pi}{4}\right)$

16. (c): $I = \int \left(\dfrac{1}{x}\right) \dfrac{1-x}{\sqrt{1-x^2}}\,dx = \int \dfrac{dx}{x\sqrt{1-x^2}} - \int \dfrac{dx}{\sqrt{1-x^2}}$

In the first integral, put $x = \dfrac{1}{t}$, we get

$I = \displaystyle\int \frac{(-1/t^2)\,dt}{\frac{1}{t}\sqrt{1 - \frac{1}{t^2}}} - \sin^{-1} x = -\int \frac{dt}{\sqrt{t^2 - 1}} - \sin^{-1} x$

$= -\log|t + \sqrt{t^2 - 1}| - \sin^{-1} x + C$

$= -\log\left|\dfrac{1}{x} + \dfrac{\sqrt{1-x^2}}{x}\right| - \sin^{-1} x + C$

$= \log|x| - \log|1 + \sqrt{1-x^2}| - \sin^{-1} x + C$

17. (b): $f(x) = \tan^{-1} x + \ln\sqrt{1+x} - \ln\sqrt{1-x}$

$\Rightarrow \quad f'(x) = \dfrac{1}{1+x^2} + \dfrac{1}{2(1+x)} + \dfrac{1}{2(1-x)}$

$= \dfrac{1}{1+x^2} + \dfrac{1}{1-x^2} = \dfrac{2}{(1-x^4)}$

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$$\Rightarrow \frac{1}{2}f'(x)=\frac{1}{1-x^4} \Rightarrow \int\frac{1}{2}f'(x)dx^4=\int\frac{dx^4}{1-x^4}$$

$$=-\ln(1-x^4)+c$$

18. (d): Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16}-\int_1^{16}\frac{x}{\sqrt{x}}\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}dx$$

$$=\frac{16}{3}\pi-\frac{1}{4}\int_1^{16}\frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$=\frac{16}{3}\pi-\frac{1}{4}\int_0^{\sqrt{3}}\frac{4t(1+t^2)}{t}dt \qquad (\text{Put } \sqrt{x}=1+t^2)$$

$$=\frac{16}{3}\pi-\left(\sqrt{3}+\sqrt{3}\right)=\frac{16}{3}\pi-2\sqrt{3}$$

19. (c): Let $g(x)=k^2x^2+\int_0^x f(t)dt-a$ ...(i)

Since, $a>0$ and $\int_0^x f(t)dt\to\infty$ as $x\to\pm\infty$ (given)

$$\Rightarrow \quad g(0)=0+\int_0^x f(t)dt-a=-a<0$$

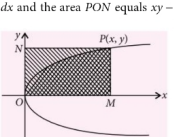
and $g(\infty)=\infty+\infty-a=\infty>0$

Also, $g(-\infty)=\infty+\infty-a=\infty>0$

$\Rightarrow$ $g(x)$ is continuous in $(-\infty,\infty)$.

Hence, $g(x)=0$ has odd number of roots in $(-\infty,0)$ and odd number of roots in $(0,\infty)$.

20. (b): Let $P(x, y)$ be the point on the curve passing through the origin $O(0, 0)$, and let PN and PM be the lines parallel to the x-axes and y-axes, respectively. If the equation of the curve is $y = y(x)$, the area POM equals $\int_0^x y\,dx$ and the area PON equals $xy-\int_0^x ydx$



Assuming that $2(POM) = PON$, we have

$$2\int_0^x y\,dx=xy-\int_0^x y\,dx \Rightarrow 3\int_0^x y\,dx=xy.$$

Differentiating both sides, we get

$$3y=x\frac{dy}{dx}+y \Rightarrow 2y=x\frac{dy}{dx} \Rightarrow \frac{dy}{y}=2\frac{dx}{x}$$

On integrating both sides, we get

$\Rightarrow \log|y|=2\log|x|+\log C \Rightarrow y=Cx^2$, with C being a constant.

This solutions represents a parabola. We will get a similar result if we had started instead with $2(PON) = POM$.

21. (d): Given curve is

$y=x^4-2x^3+x^2+3$

$$\therefore \quad \frac{dy}{dx}=4x^3-6x^2+2x$$

$$\frac{d^2y}{dx^2}=12x^2-12x+2$$

For maxima and minima, put $\frac{dy}{dx}=0$

$$\therefore \quad 4x^3-6x^2+2x=0$$

$$\Rightarrow \quad x=0,\frac{1}{2},1$$

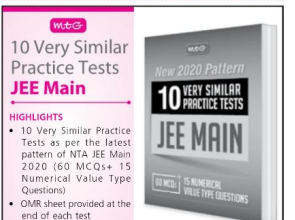
$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{x=0}=2,\ \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}}=-1 \text{ and } \left.\frac{d^2y}{dx^2}\right|_{x=1}=2$$

$\therefore$ Points of minima are $x = 0$ and $x = 1$

$$\therefore \quad \text{Required area}=\int_0^1(x^4-2x^3+x^2+3)\,dx$$

$$=\left[\frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^3}{3}+3x\right]_0^1$$

$$=\frac{1}{5}-\frac{2}{4}+\frac{1}{3}+3=\frac{1}{5}-\frac{1}{2}+\frac{1}{3}+3=\frac{91}{30}\text{ sq.units}$$



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22. (a): Putting $\frac{dy}{dx} = P$ in given equation, we get

$y = xP + \frac{1}{P}$ ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{dy}{dx} = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx}$

$\Rightarrow \quad P = P + x\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx} \qquad \left\{\because \frac{dy}{dx} = P\right\}$

$\Rightarrow \quad \frac{dP}{dx} = 0 \text{ or } P^2 = \frac{1}{x} \Rightarrow P = C \text{ or } \left(\frac{dy}{dx}\right)^2 = \frac{1}{x}$

Put these value in given equation we get $y = Cx + \frac{1}{C}$, which is equation of family of straight lines.

and $y^2 = \left(Px + \frac{1}{P}\right)^2 = P^2x^2 + 2x + \frac{1}{P^2}$

Put $P^2 = \frac{1}{x}$, we get

$y^2 = x + 2x + x = 4x$, which represents a parabola.

23. (a): Here,

Distance from focus = distance from directrix

$\Rightarrow \quad x^2 + y^2 = (2a + x)^2$

$\Rightarrow \quad y^2 = 4a(a + x)$ ...(i)

$\Rightarrow \quad 2y\frac{dy}{dx} = 4a(0 + 1)$

$\Rightarrow \quad a = \frac{y}{2}\frac{dy}{dx}$

Using (i), $y^2 = 2y\frac{dy}{dx}\left(\frac{y}{2}\frac{dy}{dx} + x\right)$

$\Rightarrow \quad y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$

24. (d): $\frac{z_1 - z_2}{z_1 + z_2} = \cos\alpha + i\sin\alpha$

$\Rightarrow \quad \frac{2z_1}{-2z_2} = \frac{\cos\alpha + i\sin\alpha + 1}{\cos\alpha + i\sin\alpha - 1}$

$= \frac{2\cos^2(\alpha/2) + 2i\sin(\alpha/2)\cos(\alpha/2)}{2i\sin(\alpha/2)\cos(\alpha/2) - 2\sin^2(\alpha/2)}$

$= \frac{2\cos(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}{2i\sin(\alpha/2)[\cos(\alpha/2) + i\sin(\alpha/2)]}$

$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{1}{i}\cot\frac{\alpha}{2} \Rightarrow iz_1 = -\cot\frac{\alpha}{2}z_2$

$\Rightarrow \quad iz_1 = Kz_2 \Rightarrow K = -\cot\alpha/2 \Rightarrow \tan\alpha/2 = -1/K$

Now, $\tan\alpha = \frac{2\tan\alpha/2}{1 - \tan^2\alpha/2} = \frac{-2/K}{1 - 1/K^2} = \frac{-2K}{K^2 - 1}$

$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2K}{1 - K^2}\right) = 2\tan^{-1}(K)$

25. (c): We have $1 + \alpha + \alpha^2 + \ldots\ldots\ldots + \alpha^{n-1} = 0$.

$\Rightarrow \quad \sum_{r=0}^{n-1}\alpha^r = 0$ ...(i)

Now, $\sum_{r=0}^{n-1}|Z_1 + \alpha^r Z_2|^2 = \sum_{r=0}^{n-1}(Z_1 + \alpha^r Z_2)(\bar{Z}_1 + \bar{\alpha}^r\bar{Z}_2)$

$= \sum_{r=0}^{n-1}|Z_1|^2 + Z_1\bar{Z}_2\sum_{r=0}^{n-1}\bar{\alpha}^r + \bar{Z}_1 Z_2\sum_{r=0}^{n-1}\alpha^r + \sum_{r=0}^{n-1}|Z_2|^2|\alpha|^{2r}$

$= n(|Z_1|^2 + |Z_2|^2)$, [Using (i) and $|\alpha| = 1$]

26. (b): Let E = Event that A gets six

$\therefore \quad P(E) = \frac{1}{6}$

and F = Event that B gets six

$\therefore \quad P(F) = \frac{1}{6}$

$\therefore \quad P(B \text{ wins}) = P(\bar{E}\,\bar{F} \text{ or } \bar{E}\,\bar{F}\,\bar{E}\,\bar{F} \text{ or } \bar{E}\,\bar{F}\,\bar{E}\,\bar{F}\,\bar{E}\,\bar{F}\ldots)$

(Since B can win the game in 2^{nd}, 4^{th}, 6^{th} …… throw)

$= \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^3\frac{1}{6} + \left(\frac{5}{6}\right)^5\frac{1}{6} + \ldots\ldots$

$= \frac{5}{36}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^4 + \ldots\right) = \frac{5}{36}\left(\frac{1}{1 - \frac{25}{36}}\right) = \frac{5}{11}$

27. (a): The given system of equations will be consistent if

$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, we get

$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & -5+3\lambda \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (5 + \lambda) + (2 + \lambda)(3\lambda - 5) = 0$

$\Rightarrow \quad 5 + \lambda + 6\lambda - 10 + 3\lambda^2 - 5\lambda = 0$

$\Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0 \Rightarrow (3\lambda + 5)(\lambda - 1) = 0$

$\Rightarrow \quad \lambda = -5/3 \text{ or } \lambda = 1.$

28. (b): For the system of equations having no solution, we must have

$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$

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$\Rightarrow\ (k+1)(k+3) = 8k$ and $8(3k-1) \neq 4k(k+3)$

But $(k+1)(k+3) = 8k \Rightarrow k^2 + 4k + 3 = 8k$

or $k^2 - 4k + 3 = 0$ or $(k-1)(k-3) = 0 \Rightarrow k = 1$ or 3

For $k = 1$, $8(3k-1) = 16$ and $4k(k+3) = 16$

$\therefore\ \ 8(3k-1) = 4k(k+3)$ for $k = 1$

For $k = 3$, $8(3k-1) = 64$ and $4k(k+3) = 72$

i.e. $8(3k-1) \neq 4k(k+3)$ for $k = 3$.

Thus, there is just one value for which the given system of equations has no solution.

29. (b): $f(A) = I + A + A^2 + \ldots\ldots + A^{16}$

Since, $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Similarly, $A^4 = A^5 = \ldots\ldots = A^{16} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$f(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \ldots\ldots + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

30. (c): $\vec{a} \perp (\vec{b} + \vec{c}) \ \Rightarrow\ \vec{a} \cdot (\vec{b} + \vec{c}) = 0$

$\Rightarrow\ \ \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0$

Also, $\vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{a} = 0$ and $\vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0$

Adding, $2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$

Now $|\vec{a} + \vec{b} + \vec{c}|^2 = (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c})$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

$= 9 + 16 + 25 + 0 = 50$

$\therefore\ \ |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$

31. (b): Lines are coplanar if $\begin{vmatrix} 1 & 1 & 1 \\ 2 & x_1 & x_2 \\ 3 & 4 & 5 \end{vmatrix} = 0$

$\Rightarrow\ \ 2x_1 - x_2 = 2$

Also, sum of roots $= \dfrac{-(x_2 + 2)}{x_1} = \dfrac{-2x_1}{x_1} = -2$

32. (a): Variance $= (\text{S.D.})^2 = \dfrac{1}{n}\Sigma x^2 - \left(\dfrac{\Sigma x}{n}\right)^2$,

$\left(\because\ \ \bar{x} = \dfrac{\Sigma x}{n}\right)$

$= \dfrac{n(n+1)(2n+1)}{6n} - \left(\dfrac{n(n+1)}{2n}\right)^2 = \dfrac{n^2 - 1}{12}$.

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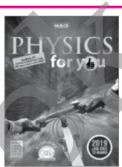

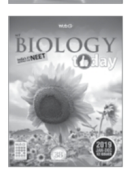

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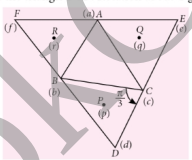
Do you have a question that you just can't get answered?

Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. Let ABC be a triangle. Construct equilateral triangles on the sides BC, CA, AB all externally or all internally. If P, Q, R are the centroids of these equilateral triangles, show that the triangle PQR is equilateral. *(Shivani, Delhi)*

Ans. ABC is any triangle, BCD, CAE, ABF are equilateral triangles. P, Q, R are the centroids of the equilateral triangles. Let the complex numbers $a, b, c, d, e, f, p, q, r$ represent the points A, B, C etc.

Consider the triangle BCD. Rotation about C gives



$d - c = (b - c)\,\alpha$, where $\alpha = e^{i\frac{\pi}{3}}$

$\therefore \quad d = b\alpha + c(1 - \alpha)$

P is the centroid

$\therefore \quad 3p = b + c + d$

or $\quad 3p = (1 + \alpha)\, b + (2 - \alpha)\, c \qquad ...(1)$

Likewise we get

$3q = (1 + \alpha)\, c + (2 - \alpha)\, a \qquad ...(2)$

$3r = (1 + \alpha)\, a + (2 - \alpha)\, b \qquad ...(3)$

From (1) to (3), we find

$3(r - p) - 3\alpha\,(q - p) = (1 + \alpha)\, a + (1 - 2\alpha)\, b - (2 - \alpha)\, c - \alpha[(2 - \alpha)a - (1 + \alpha)b + (2\alpha - 1)c]$

$= (1 - \alpha + \alpha^2)\,(a + b - 2c) = 0$

[since $1 - \alpha + \alpha^2 = 1 - e^{i\pi/3} + e^{2\pi i/3} = 0$]

$\therefore \quad r - p = \alpha\,(q - p)$

$\Rightarrow \quad |r - p| = |\alpha|\,|q - p| = |q - p|.$

Likewise $|p - q| = |r - q|$

$\therefore \quad |p - q| = |q - r| = |r - p|$

$\Rightarrow \quad \Delta PQR$ is equilateral.

2. Let the complex numbers a, b, c correspond to the points A, B, C respectively on the circle $|z| = r$. Show that the line joining A and B and the tangent to the circle at C meet at point denoted by the complex number $z = \dfrac{a^{-1} + b^{-1} - 2c^{-1}}{a^{-1}b^{-1} - c^{-2}}$. *(Manju Gupta, Patna)*

Ans. Let the chord AB and the tangent at C meet at P denoted by z.

$r^2 = a\bar{a} = b\bar{b} = c\bar{c}$. A, B, P are collinear.

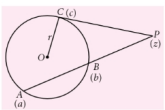
$\dfrac{z - a}{b - a} = \dfrac{\bar{z} - \bar{a}}{\bar{b} - \bar{a}}$ since $\dfrac{z - a}{b - a}$ is real

$\Rightarrow \quad (z - a)\,(\bar{b} - \bar{a}) - (\bar{z} - \bar{a})\,(b - a) = 0$

$\bar{z}\,(a - b) = z\,(\bar{a} - \bar{b}) + a\bar{b} - \bar{a}b$

$= z\left(\dfrac{1}{a} - \dfrac{1}{b}\right)r^2 + \left(\dfrac{a}{b} - \dfrac{b}{a}\right)r^2$

$\therefore \quad \bar{z} = -\dfrac{zr^2}{ab} + \dfrac{(a + b)r^2}{ab} \qquad ...(1)$



$CP^2 = OP^2 - OC^2$

$\Rightarrow \quad |z - c|^2 = |z|^2 - r^2$

$\Rightarrow \quad (z - c)\,(\bar{z} - \bar{c}) = z\,\bar{z} - r^2$

$\Rightarrow \quad c\,\bar{z} = -\bar{c}\,z + 2r^2$

$\therefore \quad \bar{z} = -\dfrac{zr^2}{c^2} + \dfrac{2r^2}{c} \qquad ...(2)$

From (1) & (2), we have $-\dfrac{z}{ab} + \dfrac{a + b}{ab} = -\dfrac{z}{c^2} + \dfrac{2}{c}$

$\Rightarrow \quad z\left(\dfrac{1}{ab} - \dfrac{1}{c^2}\right) = \dfrac{a + b}{ab} - \dfrac{2}{c} \Rightarrow z = \dfrac{a^{-1} + b^{-1} - 2c^{-1}}{a^{-1}b^{-1} - c^{-2}}.$

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